Int G&T
Connor's talk
Sep 6, 2019

Char. classes

① chern class for cx vector bundles

$$Gr^k(\mathbb{C}^\infty) = \left\{ \begin{array}{l} \text{cx } k\text{-planes} \\ \text{in } \mathbb{C}^\infty \\ = \mathbb{C} \oplus \cdots \end{array} \right\}$$

$$\begin{array}{c} E^n \\ \downarrow \\ X \end{array} \longrightarrow [X, Gr^n(\mathbb{C}^\infty)] \quad \leftarrow \text{homotopy classes of maps}$$

Thm: $H^*(Gr^k(\mathbb{C}^\infty), \mathbb{Z}) \underset{\text{as a ring}}{\cong} \mathbb{Z}[\underbrace{c_1^k, c_2^k, \ldots c_k^k}_{\text{generators}}]$

$c_0^k = 1$ $\qquad |c_i^k| = 2i$

def c_i of a k-dim cx v. bundle

$\xi : E \to X$ $\qquad \underbrace{\xi^*}_{\substack{\text{pullback} \\ \text{by the classifying} \\ \text{map}}}(c_i^k)$

Prop:
if $\xi = \varphi \oplus \underset{\substack{\text{triv.} \\ \text{bundle}}}{[k]}$

$c(\xi) = c(\varphi)$

Proof $Gr^k(\mathbb{C}^\infty) \hookrightarrow Gr^{k+1}(\mathbb{C}^\infty)$

$V \longmapsto 1 \oplus V$

$\mathbb{C}^\infty \xrightarrow{i} \mathbb{C}^\infty$

$c \, \mathbb{C} \, \mathbb{C} + \ldots$ $\qquad c_*(c_{k+1}^n) = c_k^n$.

$\xi : X \to Gr^{\dim\varphi + k}(\mathbb{C}^\infty)$

$\searrow_\varphi \; Gr^{\dim\varphi}(\mathbb{C}^\infty) \nearrow$ $\qquad \cdots$ $\qquad \square$

Euler class $\qquad$ real, oriented vect bundles

def orientation of V is a choice of a generator for $\boxed{H^{\dim V}(V, V-0) \cong \mathbb{Z}}$

def an oriented (real) vect. bundle is a bundle $\xi : E \to X$ is a choice of consistent orientation for the fibers

Thm: given an oriented vect. bundle $p^n : E \to X$, $\exists!$ $u \in H^n(E, E-X)$ so that the map induced by $(F_x, F_x - \{x\}) \overset{i}{\hookrightarrow} (E, E-X)$

then $i^*(u)$ is the generator of choice for F_x.

def The Euler class $e(p: \underset{\text{oriented}}{E} \to X)$ is defined to be the image under restriction

$j: X \hookrightarrow E$ as a zero-section

$(X, \emptyset) \hookrightarrow (E, E-X)$ inclusion of pairs

$j^*(u) \in H^n(X)$.

Euler class!

* For a non-or. bundle, Can define Euler class with coeff in $\mathbb{Z}_2$ = top S-W class.

• need X to be compact (?)

Prop the opposite orientation has opposite (= negative) Euler class

Cor: $\underset{\text{odd-dim v.b.}}{p^{2k+1}: E \to X} \Rightarrow \boxed{e(p) + e(p) = 0}$

→ use the antipodal map for a vector bundle

$\underset{\text{antipodal}}{T: E \to E^{op}}$ - for odd-dim v.b

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Prop: If $p: E \to X$ has a nowhere vanishing section, then $e(p) = 0$

Stiefel-Whitney classes

Prop: If $\xi: E \to X$, so that $\xi = \varphi + [k]$, then $w(\xi) = w(\varphi)$

proof: note $w([k]) = 1$ since $[k] = f^*\left(\begin{smallmatrix}\mathbb{R}^k \\ \downarrow \\ *\end{smallmatrix}\right)$

we deduce $w(\xi) = w(\varphi) \cup \underbrace{w([k])}_{=1} = w(\varphi)$ ☒

$M \xhookrightarrow{i} N$

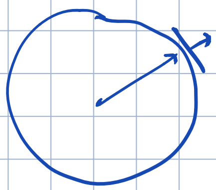
$TM \oplus \underbrace{\nu(i)}_{\text{normal bundle}} = TN|_M$

Prop $w(S^n) = 1$

Pf: consider $S^n \xhookrightarrow{i} \mathbb{R}^{n+1}$ as a unit sphere

$TR^{n+1}|_{S^n} = TS^n \oplus \nu(i)$

$$\rightarrow \quad \underbrace{w(T\mathbb{R}^{n+1}|_{S^n})}_{i^*1 \,=\, 1} = w(TS^n) \cup w(\nu(i))$$



we have a nonvanishing section $x \longmapsto \underset{\substack{\cap \\ T_xS^n}}{x}$ $\Rightarrow$ $w(\nu(i)) = 1$

$$\Rightarrow \boxed{w(TS^n) = 1}$$

TS^{2k} is a <u>nontrivial bundle</u>