An Overview of Curvature

Let (M^n, g) be a Riemannian manifold of dimension n. In order to state the Einstein field equations, we will need to review/introduce the various notions of curvature one can associate to a Riemannian manifold. This discussion uses the notation found in do Carmo's book and ideas will generally be presented in the same order as in the book as well.

Definition: Let ∇ be the Riemannian connection on M (the unique connection such that $X\langle Y, Z\rangle_g = \langle \nabla_X Y, Z\rangle_g + \langle Y, \nabla_X Z\rangle_g$ and $[X, Y] = \nabla_X Y - \nabla_Y X$ for all $X, Y, Z \in \Gamma(TM)$ i.e. compatible with g and symmetric). The Riemannian curvature of M is the covariant bilinear 4-index tensor

$$R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z$$

Note that this is just the definition of curvature for connections on vector bundles over M, where we consider TM as the vector bundle over M in question and the Riemannian connection on it. Equivalently, we can consider this definition locally in a coordinate system $\{x_i\}$ near a point $x \in M$ and see that

$$R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} = \left(\nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_j}} - \nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_i}}\right)\frac{\partial}{\partial x_k}$$

so locally, curvature measures the noncommutativity of the covariant derivative. One can check that curvature is bilinear in X, Y and linear in Z.

Remark: From now on, we will use the notation $\frac{\partial}{\partial x_i} := \partial_i$. We shall also write (X, Y, Z, T) in place of $\langle R(X, Y)Z, T\rangle_g$. Using this, we write

$$R(\partial_i, \partial_j)\partial_k = \sum_\ell R_{ijk}^\ell \partial_\ell$$

for the local expression for R. We will also write $R_{ijk\ell} = \langle R(\partial_i, \partial_j)\partial_k, \partial_\ell\rangle_g$.

Definition: Let $\sigma \subset T_pM$ be a two-dimensional subspace of TM and let x, y be two linearly independent vectors in σ. The sectional curavture of σ at p is defined by

$$\kappa(x, y) = \frac{(x, y, x, y)}{\sqrt{|x|_g^2 |y|_g^2 - \langle x, y\rangle_g}}$$

One can check that this definition of sectional curvature is actually independent of the choice of $x, y \in \sigma$ using the bilinearity of R to pass between bases of σ. Also note that the quantity in the denominator is the area of the parallelogram spanned by x, y (with respect to g). Geometrically, then, sectional curvature represents the Gaussian curvature of a surface with tangent plane σ at p which is locally the image of σ under $\exp_p$.

Definition: Let x be a vector in T_pM and let $\{z_i\}$ be an orthonormal basis of the hyperplane in T_pM which is orthogonal to x. We define the Ricci curvature (at p in the direction of x) by

$$\mathrm{Ric}_p(x) = \frac{1}{n-1}(x, z_i, x, z_i)$$

and we define the scalar curvature at p by

$$K(p) = \frac{1}{n}\,\mathrm{Ric}_p(z_i)$$

Geometrically, Ricci curvature represents how much the volume of a small conical piece of a geodesic ball around p deviates from the volume a small conical piece of a Euclidean ball. Similarly, scalar curvature represents how much the volume of a geodesic ball around p deviates from the volume of a Euclidean ball.

Remark: We will use the following notation to relate Ricci curvature and regular curvature. The terms on the left-hand side denote the terms of the Ricci curvature.

$$R_{ik} = \sum_j R^j_{ijk} = \sum_{s,j} R_{ijks} g^{sj}$$

where g^{sj} is the inverse matrix of g_{sj}. Note that this notation is valid regardless of our choice of orthonormal basis for T_pM.

The Energy-Momentum Tensor

Recall that we define momentum of an object by $p = mv$ where m is mass and v is velocity. We would like to know how momentum flows through space. In physics, this notion is called flux - the flow per unit area of a quantity. We would like to consider the flow per unit area of momentum - a vector quantity. Note that momentum encodes the force an object exerts since by Newton's second law, $F = m\dot{v}$. From this, we can see that if we consider the flow of momentum in a direction x^j through a surface with constant x^j-coordinate, then this quantity will tell us how much force is being exerted per unit area on this surface - the pressure on the surface. A related notion is the notion of shear stress which is like pressure, but in a direction coplanar with our surface. Together, shear stress and pressure form a tesnor known as the stress tensor.

The energy-momentum tensor (also known as the stress-energy tensor) T is a symmetric 2-tensor which describes the flux of momentum at a point in space. More precisely, the component T^{ij} gives the flux of the i-th component of momentum across a surface with constant j-th coordinate. If we write

$$T^{ij} = \begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix}$$

Here we have the three spatial variables x^1, x^2, x^3, and the time variable x^0. From the physical interpretation of T^{ij}, we then see that T^{00} represents the flux of momentum with respect to time, which is simply energy density (divided by c^2), also known as the relativistic mass. Similarly, the components $T^{0i} = T^{i0}$ where $i \neq 0$ represent the energy density of the i-th component of momentum. The components T^{ii} with $i \neq 0$ are the components of pressure, and the components T^{ij} with $i \neq j$ and $i, j \neq 0$ are the components of shear stress. Thus, the energy-momentum tensor represents how matter reacts to the presence of energy.

Introducing the Einstein Field Equations

We are now ready to state the Einstein field equations. These are a system of equations of the form

$$R_{ij} - \left(\frac{1}{2}K - \Lambda\right) g_{ij} = \frac{8\pi G}{c^4} T^{ij}$$

where Λ is the cosmological constant, c is the speed of light, G is the universal gravitational constant, and T^{ij} is the energy-momentum tensor. The physical meaning behind these equations is that they tell us how the curvature of spacetime is affected by energy.

A solution of the Einstein equations is a metric on spacetime. In general, it is difficult to find solutions without further assumptions on the energy-momentum tensor or on the Ricci curvature of g. One important class of solutions to the Einstein field equations is the class of so-called exact solutions. These are pseudo-Riemannian manifolds whose metric has $n-1$ positive eigenvalues and one zero eigenvalue known as Lorentzian manifolds (whereas a Riemannian metric must be positive-definite). Exact solutions include those which make various assumptions on the T^{ij}, for example that it is zero or that it arises from an electromagnetic field.

Example: One solution of the Einstein equations is the Schwarzschild metric. The metric is a spherically-symmetric Lorentzian metric and it describes the gravitational field around a spherical mass (i.e a star or a black hole). It uses the assumptions that the cosmological constant, angular momentum, and electric charge of the mass are zero. The metric is defined on $\mathbb{R} \times (0, \infty) \times \mathbb{S}^2 \simeq \mathbb{R} \times (\mathbb{E}^3 \setminus O)$ and is given in local coordinates (t, r, θ, ϕ) by

$$g = -\left(1 - \frac{r_s}{r}\right) c^2 dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi)$$

where $r_s = \frac{2GM}{c^2}$ for mass M and gravitational constant G. The physical interpretation of r_s is that this is the Schwarzschild radius for a black hole of mass M. This radius tells us the event horizon of a black hole - the point beyond which events happening outside of the black hole do not affect things near the black hole.

One can check that both the Ricci and scalar curvatures are zero for this metric and that this metric is spherically symmetric.

The Yamabe Problem

A related problem which arises from general relativity is the Yamabe problem. The problem is the following: given a smooth compact Riemannian manifold (M^n, g) with $n \geq 3$, does there exist a metric $\tilde{g}$ on M which is conformal to g and which has constant scalar curvature?

The problem is named after Hidehiko Yamabe, who claimed to have solved the problem in 1960. It was later discovered that his solution contained serious errors, and the problem was solved completely in 1984 by Trudinger-Aubin-Schoen. In the case of a non-compact smooth complete Riemannian manifold, counterexamples due to Jin exist.

References

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