INTERMEDIATE GEOMETRY AND TOPOLOGY EXERCISE SHEET 1, OCTOBER 2019

1. Chern-Simons 3-form

Fix $G = SU(2)$ and $\mathfrak{g} = \mathfrak{su}(2)$ the corresponding Lie algebra.

(a) Show that, for $A \in \Omega^1(N, \mathfrak{g})$ a connection 1-form in a trivial G-bundle over a manifold N, one has

$$\frac{1}{2}\mathrm{tr}\, F_A \wedge F_A = d\, \underbrace{\mathrm{tr}\left(\frac{1}{2} A \wedge dA + \frac{1}{6} A \wedge [A, A]\right)}_{\omega_{CS}} \tag{1}$$

where $F_A = dA + \frac{1}{2}[A, A] \in \Omega^2(N, \mathfrak{g})$ is the curvature. The 3-form ω_{CS} appearing on the r.h.s. is called the Chern-Simons 3-form.

(b) [Optional.] Find the Chern-Simons $(2k-1)$-form ω_{CS}^{2k-1} – a differential polynomial in A satisfying

$$\frac{1}{k!}\mathrm{tr}\,(F_A)^{\wedge k} = d\, \omega_{CS}^{2k-1}$$

(c) For M an oriented closed 3-manifold, consider the "Chern-Simons action functional" – a function on the space of connections in the trivial principal G-bundle on M given by

$$S_{CS}(A) = \frac{1}{(2\pi)^2} \int_M \mathrm{tr}\left(\frac{1}{2} A \wedge dA + \frac{1}{6} A \wedge [A, A]\right) \tag{2}$$

For $A^g = gAg^{-1} + gdg^{-1}$ the connection acted on by a gauge transformation (automorphism of the principal bundle) given by $g : M \to G$, prove that[1]

$$S_{CS}(A^g) - S_{CS}(A) \quad \in \mathbb{Z} \tag{3}$$

(d) In the setting of (c), let A be a connection 1-form on M. Show that the directional derivative

$$\left.\frac{d}{dt}\right|_{t=0} S_{CS}(A + t\alpha)$$

vanishes for arbitrary $\mathfrak{g}$-valued 1-form α if and only if the connection A is *flat*.

[1] Hint: use (a). An oriented closed 3-manifold M is necessarily null-cobordant, i.e. is a boundary of some compact 4-manifold N_+. Extend A to a connection A_+ over N_+ and extend A^g to a connection A_- over N_- – a copy of N_+ with opposite orientation. Then A_+ and A_- glue into a connection $\tilde{A}$ in a bundle $\tilde{\mathcal{P}}$ over the closed 4-manifold $N = N_+ \cup_M N_-$ defined by a transition function g over (a thickening of) M. Finally, compute $\frac{1}{8\pi^2}\int_N \mathrm{tr}\, F_A \wedge F_A$: on the one hand it reduces by Stokes' and (1) to the l.h.s. of (3). On the other hand, it is the Chern-Weil representative of the second Chern class of $\tilde{\mathcal{P}}$ paired with the fundamental class of N. Hence, it is an integer!

2. Lagrangian Grassmanian

For a symplectic vector space V, consider the space $\mathrm{LG}(V)$ of Lagrangian vector subspaces of V – the "Lagrangian Grassmanian."

(a) Assuming $\dim V = 2n$, find the dimension of $\mathrm{LG}(V)$. Show that $\mathrm{LG}(V)$ is a smooth compact manifold.
(b) Describe explicitly $\mathrm{LG}(\mathbb{R}^2)$, with $\mathbb{R}^2$ the standard symplectic plane.
(c) Show that, for $V = \mathbb{R}^{2n}$, the Lagrangian Grassmanian can be identified with the homogeneous space $U(n)/O(n)$.[2]

3. Moduli spaces of flat connections (flat bundles)

Recall that, for M a connected manifold and G a Lie group, the moduli space of flat bundles has the following realization:

$$\mathcal{M}_{\text{flat}}(M, G) = \mathrm{Hom}(\pi_1(M), G)/G \tag{4}$$

where Hom stands for group homomorphisms and quotient by means passing to the orbits of the action of G on homs induced by action of G on (target) G by conjugation. The r.h.s. is understood as the set of orbits equipped with quotient topology.

(a) Show that $\mathcal{M}_{\text{flat}}(S^1, G) = G/G$ is the set of conjugacy classes of G. Show that for $G = SU(2)$, one gets

$$\mathcal{M}_{\text{flat}}(S^1, SU(2)) = \frac{\{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \mid \theta \in \mathbb{R}/2\pi\mathbb{Z}\}}{\theta \sim -\theta}$$

Thus, points of $\mathcal{M}_{\text{flat}}(S^1, SU(2))$ are in bijection with points on the interval $[0, \pi] \ni \theta$, where the isotropy subgroup/stabilizer (the subgroup of G leaving invariant the given homomorphism $\pi_1(S^1) \to G$) is $U(1)$ (diagonal matrices of unit determinant) for $0 < \theta < \pi$ and is the entire $SU(2)$ for $\theta = 0, \pi$.
(b) Find explicitly $\mathcal{M}_{\text{flat}}(\text{Klein bottle}, SU(2))$.[3]
(c) Show that, for Σ a closed oriented surface of genus g, the moduli space with coefficients in $U(1)$ is a $2g$-torus:

$$\mathcal{M}_{\text{flat}}(\Sigma, U(1)) = \underbrace{S^1 \times \cdots \times S^1}_{2g}$$

[2] The solution in on p.22 in `https://math.berkeley.edu/~alanw/GofQ.pdf`.

[3] Answer: $\mathcal{M}_{\text{flat}}(\text{Klein bottle}, G) = \frac{\{(x,y)\in G \mid y^{-1}=xyx^{-1}\}}{(x,y)\sim(gxg^{-1},gyg^{-1})}$. For $G = SU(2)$, the equivalence classes in $\mathcal{M}_{\text{flat}}$ can be represented by pairs $(x = e^{i\theta\sigma_1}, y = e^{i\phi\sigma_3})$ where $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and the admissible values for θ, ϕ are: $(\phi, \theta) \in [0, \pi] \times \{\frac{\pi}{2}\} \cup \{0\} \times [0, \pi] \cup \{\pi\} \times [0, \pi]$. What are the isotropy subgroups for different equivalence classes here?