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Geom Quant of the moduli space of flat connections

1. motivation

Recall Kähler polarization

(M,ω) cpt conn Kähler mfld

$I \in \mathrm{Aut}(TM)$

(1) $I^2 = -1$

(2) $g(X,Y) = \omega(IX,Y)$ is a metric (pos. def)

(3) integrability

Newlander–Nirenberg thm: $[IX,IY] = [X,Y] + I[IX,Y] + I[X,IY]$

$X, Y \in TM$

- Assoc. to M a vect. space $V = \bigcup_{t \in T} V_t \longleftrightarrow I(t)$

T parameter space

Fact: V is a vect bundle over T

V_t is independent of $I \Rightarrow$ identify all V_t's

$V_I \cong V_J$ true $\forall$ path $I, J \Leftrightarrow$ conn. flat

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Thm (from Eric's talk) X is cpt Kähler, G a Lie grp with bi-invar metric. Then $\mathcal{M}(X,G) \cong \mathrm{Hom}(\pi_1 X, G)/G$ admits a symp structure.

2. Kähler structure on $\mathcal{M}(X,G)$.

$X = \Sigma$ cpt Kähler surface

On $\Omega^1(\Sigma)$ we have the form $(\alpha,\beta) \mapsto \int_\Sigma \alpha\wedge\beta$

nondeg skew-sym bilinear form

$A_1, A_2 \in \mathcal{A} =$ "connections over $P \to M$" , $\mathcal{G}$ is gauge group

μ moment map $\mu(A) = F_A = dA + \frac{1}{2}[A,A]$

$\mathcal{M} \cong \mu^{-1}(0)/\mathcal{G}$ - symp. reduction.

$= \mathcal{A}/\!/\mathcal{G}$

if $A_1 - A_2 \in \Omega^1(\Sigma, \operatorname{ad} P)$ $\qquad$ $\operatorname{ad} P = P \times_G \mathfrak{g}$

$\mathcal{A}$ - affine space modeled on $\Omega^1(\Sigma, \operatorname{ad} P)$.

<u>def</u> (A, V, δ) A-affine space modeled on V if $\delta : A \times A \to V$

(i) $\forall a \in A, v \in V$ $\exists!\, b \in A : \delta(a,b) = v$
(ii) $\delta(a,b) + \delta(b,c) = \delta(a,c)$

On $\mathcal{A}$:

$$\omega_{AB} = \int_\Sigma B(\alpha_1 \wedge \alpha_2) \qquad \alpha_1, \alpha_2 \in \Omega^1(\Sigma, \operatorname{ad}(P))$$

Atiyah-Bott form

B: bi-inv. metric on $\mathfrak{g}$.

ω_{AB} : nondeg skew sym (transl. invariant)

$\to$ ω is a symp. structure on $\mathcal{A}$.

Talk 2 | $\hbar = 1$

$$\mathcal{M}_{flat} \cong \mu^{-1}(0)/\mathcal{G} = \mathcal{A}/\!/\mathcal{G}$$

$$\omega_{AB} = \int_\Sigma B(\alpha_1 \wedge \alpha_2)$$

$\alpha_1, \alpha_2 \in \Omega^1(\Sigma, \mathrm{ad}\, P)$

E.g. $G = SU(m)$

$$\omega_{AB} = -\int \mathrm{tr}(\alpha_1 \wedge \alpha_2) \quad \text{(for positive-definiteness)}$$

Kähler structure on $\mathcal{A}$: Σ Kähler $\quad g_1 = e^{2f} g_2$ - conf. metrics

Hodge $*$: $\Omega^1(\Sigma) \to \Omega^1(\Sigma)$

$*^2 = -1$

Each class $[\alpha] \in H^1_A(\Sigma, \mathrm{ad}\, P)$

has a unique harmonic representative $\quad d_A \alpha = d_A * \alpha = 0$

$I\alpha = -*\alpha$

Can check $\quad -\omega(I\alpha, \alpha) = -\int B(\alpha \wedge *\alpha) > 0 \Rightarrow$ Kähler structure on $\mathcal{A}$

$$\mathcal{M}_{flat} \cong \mathcal{A}^{ss}/\mathcal{G}^{\mathbb{C}}$$

(Narasimhan–Seshadri)

$\mathcal{A}^{ss}$ (semi-stable) $= \bigcup \{ \mathcal{G}^{\mathbb{C}}$ orbit of points in $\mu^{-1}(0) \}$

E.g. $G = U(1)$

sheaf of holom. fun.

$$\mathcal{M}_{flat} \cong H^1(\Sigma, \mathcal{O}) / H^1(\Sigma, \mathbb{Z}) \cong \text{complex torus}$$

from exp sequence $\quad 0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$

Rem Kähler structure on $\mathcal{A}$ $\to$ curvature of a unitary (hermitian) conn.

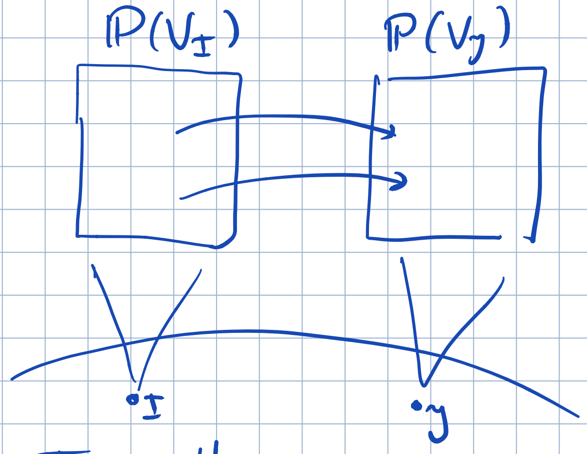
on det bundle of a family of CR operators: $\bar\partial_A : \Omega^0(\Sigma, E) \to \Omega^1(\Sigma, E)$

replace $\mathrm{ad}(P) \rightsquigarrow \mathrm{End}_0(E)$ → another line bdle

trace $= 0$

$\rightsquigarrow$ curvature $\quad \lambda\, \omega_{AB}, \quad \lambda \in \mathbb{Z}^+$

Grothendieck-Riemann-Roch



Geom Quant:

Flat connection

on $\mathbb{P}(H^0_{\bar\partial}(\mathcal{M}_{flat}, \mathcal{L}^{\otimes k}))$

k - "level" $\in \mathbb{Z}$

$T = \mathcal{M}_{cx \text{ structures}}$

$V_I = \{ s \in \Omega^0(\mathcal{M}, \mathcal{L}^{\otimes k})$
$(1 + iI)\nabla s = 0\}$

(M, ω) symplectic
$\frac{1}{2\pi}[\omega] \in H^1_{int}(\Sigma, \mathbb{R})$
$\rightsquigarrow L$ - preq. line bdle

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3. Hitchin connection

(M^{2n}, ω)

L preq. line bdle

I Kähler polarization,

$V_I = \{ s \in \Omega^0(M, L) : \underbrace{(1+iI)\nabla}_{\nabla^{0,1}} s = 0\}$

$\dim H^0_{\bar\partial}(M, L) = \dim V_I \rightsquigarrow GIT$

- I_t path of K. polar.

$\nabla_t^{0,1} s_t = 0$

$\Rightarrow \quad i \dot{I} \nabla s_t + (1 + iI) \nabla \dot{s} = 0 \quad (1)$

connection: $u(s, \dot{I})$
tangent vector to $\mathcal{M}$ cx str

$i \dot{I} \nabla s + (1 + iI) \nabla u = 0 \quad (2)$

auxiliary condition? (ansatz?)

$(1), (2) \Rightarrow \boxed{i \dot{I} \nabla^{1,0} s + \nabla^{0,1} u = 0} \quad (*)$

(Hitchin '90) local sol exists.

$\mathcal{D}^k L$: vect. bundle of hol. linear diff operators of order k.

Fact: $\quad 0 \to \mathcal{D}^{k-1}(L) \to \mathcal{D}^k(L) \xrightarrow{\sigma} S^k TM$

σ - principal symbol $\quad \sigma\left(\sum_{|\alpha|\le k} a_\alpha D^\alpha\right) \to \sum_{|\alpha|\le k} a_\alpha \xi^\alpha$

$(*) \Rightarrow \bar\partial(i\dot{I}\nabla^{1,0}) = 0 \quad \in \Omega^{0,2}(M, \mathcal{D}^1(L))$

Def $\Lambda^p = \Omega^{0,p}(M, \mathcal{D}^1(L)) \oplus \Omega^{0,p-1}(M, L)$

$d_s : \Lambda^p \to \Lambda^{p+1}$

$d_s(D, u) = (\bar\partial D, \bar\partial u + (-1)^{p-1} Ds)$

$d_s d_s = 0 \to$ chain cx

$\Rightarrow H_s^p(M, \mathcal{D}^1 L)$

In our context

$\sigma(\dot{I}\nabla^{1,0}u) = [\dot{I}] \longrightarrow$ "Kodaira-Spencer deformation class"

$\omega = \sum \omega_{i\bar j}\, dz_i \wedge d\bar z_j$

$[\omega]\left(X_i \frac{\partial}{\partial z_i}\right) = X_i \omega_{i\bar j}\, d\bar z_j$

<u>Thm</u>: (M, ω) cpt symp, $\underset{\hbar}{\overset{L}{\downarrow}}, \nabla$, $F_\nabla = \omega$,

$\{I_t\}$ family of Kähler polarizations

(i) $[\omega] : H^0(M, TM) \xrightarrow{\sim} H^1(M, \mathcal{O})$ is an iso

(ii) $\forall s \in H^0(M, L)$, tangent vector $\dot{I}$ $\exists$ A smooth $A(\dot{I}, s) \in H^1_s(M, \mathcal{D}(L))$

$-i\sigma A(\dot{I}, s) = [\dot{I}]$

$\Rightarrow A$ defines a connection $\quad \mathbb{P}(H^0(M, L)) \longrightarrow \{I_t\}$

In our case

λ

$S^2 TM_{flat} \ni G$

$$0 \to H^0_s(M, \mathcal{D}^1(L)) \to H^0_s(M, \mathcal{D}^2 L) \xrightarrow{\sigma} H^0(M, S^2TM)$$

$$\xrightarrow{\delta} H^1_s(M, \mathcal{D}^1(L)) \to \dots$$

$\dfrac{\delta(G)}{(2k+\lambda)i} \in H^1_s(M_{flat}, \mathcal{D}^1(L^{\otimes k}))$ connection

this conn. is flat!

on

10/28/19

Talk 3

Last time

Hypercohomology

$\Lambda^p = \Omega^{0,p}(M, D^1(L)) \oplus L^{0,p-1}(M,L)$

$d_s(D,u) = (\bar\partial D, \bar\partial u + (-1)^{p-1} D s)$

$\mathbb{H}^p_s(M^\cdot, D^1(L)) = \frac{\ker d_s}{\operatorname{im} d_s}$

$\hookrightarrow \quad 0 \to D^{k-1}(L) \to D^k(L) \xrightarrow{S} S^k T \to 0$

Connection rep : $\frac{\delta(G)}{2k+?}$

$T = T^{1,0}M$

4. An explicit formula.

$\mathcal{M} = \operatorname{Hom}(\pi_1(\Sigma), SU(n))/SU(n)$

$\mathcal{L} \to (\mathcal{M}, \omega)$ curvature

$\mathcal{L}^{\otimes k} \rightsquigarrow k\omega$

$\nabla_i = \nabla_{\frac{\partial}{\partial z_i}}, \qquad \nabla_{\bar j} = \nabla_{\frac{\partial}{\partial \bar z_j}}$

want: $\Delta \in C^\infty(\mathcal{D}^2(L)) \qquad \sigma(\Delta) = G$

holom. Laplacian

$d_s(\Delta, 0) = (\underbrace{\bar\partial \Delta}_{\delta^1(G)}, -\Delta s) \in H^1_s(M, \mathcal{D}^1(L))$

choose $\Delta s = \nabla_i(G^{ij}\nabla_j s)$ "usual" Laplace-Beltrami operator

$\nabla_{\bar\ell}\Delta s = (R_{\bar\ell i} + k\omega_{\bar\ell i}) G^{ij} \nabla_j s$
$\quad + \nabla_i \nabla_{\bar\ell} G^{ij} \nabla_j s$

$R_{i\bar j}$ Ricci curvature

$G^{ij} \nabla_j s \otimes \frac{\partial}{\partial z_i}$ section of $L \otimes TM$

$R_{j\bar k}\, dz_j \wedge d\bar z_k \to 1^{st}$ Chern class $\sim$ cohomologous to $\lambda\omega$

$\in S(G)$ rep. $\quad -(2k+1)\, \underbrace{G^{ij}\omega_{j\bar{k}} \frac{\partial}{\partial z_i} \otimes dz_k}_{[\dot{I}]}$

$$\boxed{u(\dot{I}, s)}$$

Solving for $u(\dot{I}, s)$

$$u(\dot{I}, s) = -\frac{i}{2k+1}\left[\nabla_i (G^{ij} \nabla_j s) - 2i\, G^{ij} \frac{\partial F}{\partial z_i} \nabla_j s + ik f_G s \right)$$

$$R_{j\bar{k}} - \lambda \omega_{j\bar{k}} = 2i \quad \partial^2 F$$