Computing N_d using intersection theory on the Kontsevich space $\overline{M}_{0,n}(\mathbb{P}^2, d)$

Fall 2019, Intermediate Geometry and Topology

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November 11, 2019

Strategy:

1. Recall that $N_d = \#$ degree-d rational plane curves passing through $3d - 1$ general points in $\mathbb{P}^2$.
2. Define a curve $Y \subseteq \overline{M}_{0,n}(\mathbb{P}^2, d)$.
3. Intersect Y with boundary components of $\overline{M}_{0,n}(\mathbb{P}^2, d)$.
4. Use linear equivalence (Keel) relations.

Set-up.

Let $n = 3d$ (not $3d - 1$).
Label the marked points by $\{1, 2, \ldots, n-4, q, r, s, t\}$.
Recall that we have the forgetful morphism

$$\overline{M}_{0,n}(\mathbb{P}^2, d) \to \overline{M}_{0,\{q,r,s,t\}}.$$

Pulling back the boundary points of $\overline{M}_{0,\{q,r,s,t\}}$ gives Keel relations in $\overline{M}_{0,n}(\mathbb{P}^2, d)$:

$$\sum_{\substack{d_1+d_2=d \\ q,r\in A \\ s,t\in B}} D(A,B;d_1,d_2) = \sum_{\substack{d_1+d_2=d \\ q,s\in A \\ r,t\in B}} D(A,B;d_1,d_2) = \sum_{\substack{d_1+d_2=d \\ q,t\in A \\ r,s\in B}} D(A,B;d_1,d_2).$$

Define the curve Y.

Let $z_1, \ldots, z_{n-4}, z_s, z_t$ be general points and let l_q, l_r be general lines in $\mathbb{P}^2$. Then

$$Y = \rho_1^{-1}(z_1) \cap \cdots \cap \rho_{n-4}^{-1}(z_{n-2}) \cap \rho_q^{-1}(l_q) \cap \rho_r^{-1}(l_r) \cap \rho_s^{-1}(z_s) \cap \rho_t^{-1}(z_t)$$

is a curve in $\overline{M}_{0,n}(\mathbb{P}^2, d)$.

Sanity check:

- $\dim \overline{M}_{0,n}(\mathbb{P}^2, d) = \dim \mathbb{P}^2 + \int_{d[\text{line}]} c_1(T\mathbb{P}^2) + n - 3 = 2 + 3d + n - 3 = 2n - 1.$
- codim $Y = 2(n-2) + 1(2) = 2n - 2$ because there are $n-2$ points (which have codim 2) and 2 lines (which have codim 1).

Intersect Y with boundary components of $\overline{M}_{0,n}(\mathbb{P}^2, d)$.

Bertini + general position of the points and lines $\implies$

- Y is a nonsingular curve in the automorphism-free locus.
- Y intersects all boundary divisors transversally at general points of the boundary.

A point in $Y \cap D(A, B; d_1, d_2)$ with $q, r \in A$ and $s, t \in B$ is represented by a pointed map $\mu : C_A \cup C_B \to \mathbb{P}^2$.

Now we split into cases to count these pointed maps.

Intersect Y with boundary components of $\overline{M}_{0,n}(\mathbb{P}^2, d)$.

<u>Case I</u>: $d_1 = 0, d_2 = d$.

- μ maps C_A to the point $l_q \cap l_r$.
- If there is some point other than q, r in A, then μ maps that point to $l_q \cap l_r$, which contradicts the assumption that the points and lines lie in general position.
- Thus $Y \cap D(A, B; 0, d) \neq \emptyset$ only when $A = \{q, r\}$.
- μ takes the $3d - 2$ points in B to the $3d - 2$ general points in $\mathbb{P}^2$.
- μ takes $C_A \cap C_B$ to $l_q \cap l_r$.
- Therefore $\# Y \cap D(\{q, r\}, \{1, \ldots, n-4, s, t\}; 0, d) = N_d$.

Intersect Y with boundary components of $\overline{M}_{0,n}(\mathbb{P}^2, d)$.

Case II: $1 \leq d_1 \leq d-1$.

- $Y \cap D(A,B;d_1,d_2) \neq \emptyset$ only when $|A| = 3d_1 + 1$ due to general position.
- There are $\binom{3d-4}{3d_1-1}$ partitions such that $q, r \in A$, $s, t \in B$ and $|A| = 3d_1 + 1$.
- For each partition, $\# Y \cap D(A,B;d_1,d_2) = N_{d_1} N_{d_2} d_1^3 d_2$:
  - $\#$ choices of $\mu(C_A)$ (discounting q, r) $= N_{d_1}$.
  - $\#$ choices of $\mu(C_B) = N_{d_2}$.
  - $\#$ choices of $\mu(q) = \#\mu(C_A) \cap l_q = \deg(\mu(C_A)) = d_1$.
  - $\#$ choices of $\mu(r) = \#\mu(C_A) \cap l_r = \deg(\mu(C_A)) = d_1$.
  - $\#$ choices of $C_A \cap C_B = \#\mu(C_A) \cap \mu(C_B)$ $= \deg(\mu(C_A)) \cdot \deg(\mu(C_B)) = d_1 d_2$.

Case III: $d_1 = d, d_2 = 0$. $Y \cap D(A,B;d_1,d_2) = \emptyset$ due to general position.

Use linear equivalence (Keel) relations.

Summing all the cases,

$$\#Y \cap D(q,r|s,t) = \sum_{\substack{d_1+d_2=d\\ q,r\in A\\ s,t\in B}} D(A,B;d_1,d_2)$$

$$= N_d + \sum_{\substack{d_1+d_2=d\\ d_1,d_2>0}} N_{d_1} N_{d_2} d_1^3 d_2 \binom{3d-4}{3d_1-1}.$$

Similar calculation gives

$$\#Y \cap D(q,s|r,t) = \sum_{\substack{d_1+d_2=d\\ d_1,d_2>0}} N_{d_1} N_{d_2} d_1^2 d_2^2 \binom{3d-4}{3d_1-2}.$$

Use linear equivalence (Keel) relations.

By Keel relations, these numbers equal, so we get the recursive formula:

$$N_d = \sum_{d_1+d_2=d} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right).$$

Note that this is the same formula obtained from computation using quantum cohomology.

Reference

Section 0.6 of “Notes On Stable Maps And Quantum Cohomology” by Fulton and Pandharipande.