LAST TIME: • $V \otimes W$

• $TV = \bigoplus_{p=0}^{\infty} V^{\otimes p}$ - tensor algebra

• $\Lambda^{\bullet} V = TV / \underset{\uparrow}{I(V)}$ $\qquad \Lambda^p V = \pi(V^{\otimes p})$

ideal generated by $\{v \otimes v \mid v \in V\}$

• $\wedge$ - product on $\Lambda^{\bullet} V$ induced from $\otimes$ - product on TV

• $\alpha \in \Lambda^p V$, $\beta \in \Lambda^q V \Rightarrow \alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha \in \Lambda^{p+q} V$

* Remark: $\overbrace{v \wedge \alpha \wedge w}^{(-1)^{p+1}} = - w \wedge \alpha \wedge v \Rightarrow \beta \wedge v \wedge \alpha \wedge w \wedge \gamma =$

$\underset{V}{v}$, $\underset{\Lambda^p V}{\alpha}$, $\underset{V}{w}$; $(-1)^p$

$= - \beta \wedge w \wedge \alpha \wedge v \wedge \gamma$

(can interchange any two vectors in a wedge product at the cost of changing the sign)

Proposition 5.3 If $\dim V = n$, then $\dim \Lambda^n V = 1$

Proof: • For $w_1, \dots, w_n \in V$, consider $B(w_1, \dots, w_n) := \det \underset{\uparrow}{M}$ - a non-zero alternating n-linear form on V

matrix with columns = coordinate vectors of $w_1, \dots w_n$ rel. to some basis in V

So, $\underset{\neq 0}{B} \in (\Lambda^n V)^*$, so $\dim \Lambda^n V \geq 1$

• if $v_1, \dots, v_n$ - basis in V, then anything in $V^{\otimes n}$ is a lin. comb. of $v_{i_1} \otimes \dots \otimes v_{i_n}$.

So, by Prop. (#), anything in $\Lambda^n V$ is a lin. comb. of $v_1 \wedge \dots \wedge v_n$.

so, $\dim \Lambda^n V \leq 1$ $\square$

Proposition 5.4 Let $v_1, \dots, v_n$ be a basis for V. Then the $\binom{n}{p}$ elements $\{v_{i_1} \wedge \dots \wedge v_{i_p}\}_{1 \leq i_1 < \dots < i_p \leq n}$ form a basis for $\Lambda^p V$.

Proof $\Lambda^p V$ is spanned by $\{v_{i_1} \wedge \dots \wedge v_{i_p}\}_{1 \leq i_1 \leq n, \dots, 1 \leq i_p \leq n} \underset{\uparrow}{\Rightarrow} \Lambda^p V = \mathrm{Span}\{v_{i_1} \wedge \dots \wedge v_{i_p}\}_{i_1 < \dots < i_p}$

reordering / changing sign

Linear independence: suppose $\sum_{i_1<\dots<i_p} a_{i_1\dots i_p} v_{i_1}\wedge\dots\wedge v_{i_p} = 0$ (@)

$$@ \wedge \boxed{\overset{\beta}{v_{j_1}\wedge\dots\wedge v_{j_{n-p}}}} \;\Rightarrow\; \pm a_{i_1\dots i_p} \underbrace{v_1\wedge\dots\wedge v_n}_{\neq 0} = 0$$

where $\underbrace{\{i_1,\dots,i_p\}}_{\text{fixed subset of } \{1\dots n\}} \cup \{j_1,\dots,j_{n-p}\} = \{1,2,\dots,n\}$

(all other terms in the lhs of @ vanish after wedging with β)

$\Rightarrow a_{i_1\dots i_p} = 0$ for any $i_1<\dots<i_p$ ☒

Proposition 5.5 v is linearly dependent on the set of lin. indep. vectors $v_1,\dots,v_p$ iff $v_1\wedge\dots\wedge v_p\wedge v = 0$

Proof: if $v = \sum_i a_i v_i$ - lin. dependent on v_i s, then

$$(v_1\wedge\dots\wedge v_p)\wedge v = \sum_i a_i v_1\wedge\dots\wedge v_p\wedge v_i = 0$$

if v is lin. indep., then the set $v_1,\dots,v_p,v$ can be extended into a basis for V and by the previous Prop. 5.4, $v_1\wedge\dots\wedge v_p\wedge v \neq 0$. ☒

def if $A: V\to W$ is a lin. transformation, then there is an induced linear transformation $\Lambda^p A: \Lambda^p V \longrightarrow \Lambda^p W$ defined by $v_1\wedge\dots\wedge v_p \mapsto Av_1\wedge\dots\wedge Av_p$ (and extended by linearity)

- Another construction of $\Lambda^p A$: by univ. property

$$\begin{tikzcd} V\times V \arrow[r, "\otimes"] \arrow[dr] & V\otimes V \arrow[d, dashed, "A^{\otimes 2}"] \\ & W\otimes W \end{tikzcd} \qquad (v_1,v_2) \mapsto Av_1\otimes Av_2$$

$$\begin{tikzcd} V\times\dots\times V \arrow[r, "\otimes"] \arrow[dr] & V^{\otimes p} \arrow[d, dashed, "A^{\otimes p}"] \\ & W^{\otimes p} \end{tikzcd} \qquad (v_1,\dots,v_p) \mapsto Av_1\otimes\dots\otimes Av_p$$

- $A^{\otimes p}(I(V)) \subset I(W)$

$\Rightarrow A^{\otimes p}$ induces a map of quotients $\Lambda^p A: \underbrace{V^{\otimes p}/I(V)}_{=\Lambda^p V} \longrightarrow \underbrace{W^{\otimes p}/I(W)}_{=\Lambda^p W}$

Proposition 5.7 If $\dim V = n$ and $A: V \to V$, then the linear transformation $\Lambda^n V \to \Lambda^n V$ is given by (multiplication by) $\det A$. ← determinant of the matrix of A rel. to some basis

Proof: $\Lambda^n V$ is 1-dimensional, so $\Lambda^n A : \Lambda^n V \to \Lambda^n V$ is a multiplication by a real number $\lambda(A)$. For $v_1, \dots, v_n$ a basis in V,

$$\Lambda^n A(v_1 \wedge \dots \wedge v_n) = \underbrace{Av_1 \wedge \dots \wedge Av_n} = \lambda(A)\, v_1 \wedge \dots \wedge v_n$$

$$Av_i = \sum_j A_{ji} v_j$$

$$\begin{aligned} Av_1 \wedge \dots \wedge Av_n &= \sum_{j_1 \dots j_n} A_{j_1 1} v_{j_1} \wedge \dots \wedge A_{j_n n} v_{j_n} \\ &= \sum_{\sigma \in S_n} A_{\sigma(1)1} v_{\sigma(1)} \wedge \dots \wedge A_{\sigma(n)n} v_{\sigma(n)} \quad \leftarrow \text{permutations} \\ &= \underbrace{\sum_{\sigma \in S_n} \operatorname{sign}\sigma \cdot A_{\sigma(1)1} \cdots A_{\sigma(n)n}}_{\det A}\; v_1 \wedge \dots \wedge v_n \end{aligned}$$

$\square$

Differential forms

The bundle of p-forms. Let M - n-manifold, T_x^* - cotangent space, $\Lambda^p T_x^*$ - its exterior power

consider $\Lambda^p T^* M := \bigcup_{x \in M} \Lambda^p T_x^*$ (with a natural projection π to M) – we will endow it with the structure of a vector bundle (and thus a manifold)

- Let (U, φ_U) a chart on M with $x_1, \dots, x_n$ coordinates.

Then, for $x \in U$, the elements $dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_p}$ for $i_1 < \dots < i_p$ form a basis of $\Lambda^p T_x^*$.

The $\binom{n}{p}$ coefficients of $\alpha \in \Lambda^p T_x^*$ then give a coord. chart Ψ_U

$$\underbrace{\bigcup_{x \in U} \Lambda^p T_x^*}_{\pi^{-1}(U)} \xrightarrow{\Psi_U} \varphi_U(U) \times \mathbb{R}^{\binom{n}{p}} \underset{\text{open}}{\subset} \mathbb{R}^n \times \mathbb{R}^{\binom{n}{p}}$$

- for $p = 1$, this is just the coord. chart we had for the cotangent bundle T^*M

$\psi_U(x, \sum_i y_i dx_i) = (x_1, \dots, x_n; y_1, \dots, y_n)$ and on an overlap $U_\alpha \cap U_\beta$ we had

$$\psi_\beta \circ \psi_\alpha^{-1}(x_1, \dots, x_n; y_1, \dots, y_n) = (\tilde{x}_1, \dots, \tilde{x}_n; \sum_j \frac{\partial x_j}{\partial \tilde{x}_1} y_j, \dots, \sum_j \frac{\partial x_j}{\partial \tilde{x}_n} y_j)$$ ← there is a mistake in Hitchin here!

- For p arbitrary, we replace the matrix $A = \underbrace{\frac{\partial x_j}{\partial \tilde{x}_i}}_{Jac^{-1T}}$ by its p-th exterior power

$\Lambda^p A : \Lambda^p \mathbb{R}^n \to \Lambda^p \mathbb{R}^n$
- complicated to write in a basis, but
- invertible
- C^∞ in x

$\Rightarrow \psi_\beta \circ \psi_\alpha^{-1}$ is C^∞ on overlaps $\Rightarrow \Lambda^p T^* M$ is a smooth manifold, $\dim = n + \binom{n}{p}$

- __def__ The bundle of p-forms on M is the smooth mfd $\Lambda^p T^* M$ with the smooth structure as above; There is a natural projection $\pi : \Lambda^p T^* M \to M$; a section is called a __differential p-form__.

__Ex:__ 1. A zero-form is a section of $\Lambda^0 T^* M$ which is by convention a smooth function f.

2. A 1-form is a section of the cotangent bundle. E.g. df is a 1-form.
In loc. coords: $df = \sum_j \frac{\partial f}{\partial x_j} dx_j$.

- By using a bump function, we can extend a locally-defined p-form like $dx_1 \wedge \ldots \wedge dx_p$ to the whole M $\Rightarrow$ sections always exist.

- one can represent functions / vector fields / p-forms as sums of local ones using a __partition of unity__.

- __Partitions of unity__

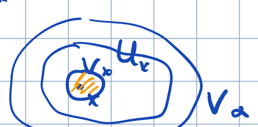
__def__ A __partition of unity__ on M is a collection $\{\varphi_i\}_{i \in I}$ of smooth functions s.t.

- $\varphi_i \geq 0$
- $\{\operatorname{supp} \varphi_i : i \in I\}$ is locally finite, i.e. $\forall x \in M$ $\exists U$ open nbhd which intersects only finitely many supports $\operatorname{supp} \varphi_i$.
- $\sum_i \varphi_i = 1$

__Theorem__ (Thm 10.8) Given any open cover $\{V_\alpha\}$ of a manifold M, there exists a partition of unity $\{\varphi_i\}$ on M s.t. $\operatorname{supp} \varphi_i \subset V_{\alpha(i)}$ for some $\alpha(i)$.

(then one says that $\{\varphi_i\}$ is "subordinate" to the cover $\{V_\alpha\}$)

__Proof in case M compact:__ $\forall x \in M$ take a coord. nbhd $U_x \subset \underset{\text{some } \alpha}{V_\alpha}$ and a bump function μ_x which is 1 in a nbhd $\underset{x}{V_x} \subset U_x$ and with $\operatorname{supp} \mu_x \subset U_x$



$\{V_x\}_{x \in M}$ is a cover of M $\underset{\text{compactness}}{\Longrightarrow}$ can find a finite subcover $\{V_{x_i}\}_{i=1\ldots N}$

Set $\varphi_i = \dfrac{\mu_{x_i}}{\sum_{j=1}^{\tilde{n}} \mu_{x_j}}$ - this is a partition of unity subordinate to $\{V_\alpha\}$ $\square$