$\Rightarrow g(x_1 \ldots x_n) = h(y_1(x), \ldots, y_n(x))$

$\Rightarrow \dfrac{\partial g}{\partial x_i} = \displaystyle\sum_j \frac{\partial h}{\partial y_j}(y(x)) \underbrace{\frac{\partial y_j}{\partial x_i}(x)}_{\substack{\text{invertible matrix,} \\ \text{since } y(x) \text{ is invertible}}}$ $\Rightarrow Dg|_{x(a)} = 0$ iff $Dh|_{y(a)} = 0$

chain rule

$\Rightarrow$ vanishing of the derivative at a is independent of the coord. chart.

Let $Z_a = \{ f \in C^\infty(M) \mid f \text{ has vanishing derivative at } a \} \subset C^\infty(M)$ vect. subspace

def The <u>cotangent space</u> T_a^* at $a \in M$ is the quotient space $T_a^* = C^\infty(M) / Z_a$. The derivative of f at $a \in M$ is its image in this space and is denoted $(df)_a$.

$f \in C^\infty(M)$

- if $f \in C^\infty(M)$, $(df)_a = d(\mu \cdot f)_a$ $\Rightarrow$ can make sense of $(df)_a$ for a locally-defined smooth f (in a nbhd of a), such as $f = x_1, \ldots, x_n$ - loc. coord. functions

μ: bump function $\equiv 1$ in the nbhd of a

since if v vanishes in nbhd of a, then $v \in Z_a \Rightarrow (dv)_a = 0$. set $v = f - \mu f$

<u>Proposition</u>: Let M be an n-mfd. Then

- The cotangent space T_a^* at $a \in M$ is an n-dimensional vector space.
- If (U, φ) is a coord. chart around a with coords $x_1, \ldots, x_n$, then the elements $(dx_1)_a, \ldots, (dx_n)_a$ form a basis for T_a^*.
- If $f \in C^\infty(M)$ and in the coord. chart, $f \circ \varphi^{-1} = \psi(x_1, \ldots, x_n)$ then

$$(df)_a = \sum_i \frac{\partial \psi}{\partial x_i}(\varphi(a))\,(dx_i)_a \qquad (*)$$

<u>Proof</u> $f - \displaystyle\sum_i \frac{\partial \psi}{\partial x_i}(\varphi(a))\, x_i$ - locally-defined smooth function whose derivative vanishes at a.

$\Rightarrow (df)_a = \displaystyle\sum \frac{\partial \psi}{\partial x_i}(\varphi(a))(dx_i)_a$

and $(dx_i)_a$ span T_a^*. If $\displaystyle\sum_i \lambda_i (dx_i)_a = 0$ then $\displaystyle\sum_i \lambda_i x_i$ has vanishing derivative at $a \Rightarrow \lambda_1 = \ldots = \lambda_n = 0$.

<u>Rem</u> We will denote $\psi = f$.

ψ: coord. representation of f

So that $(*)$ becomes: $df = \displaystyle\sum_i \frac{\partial f}{\partial x_i} dx_i$.

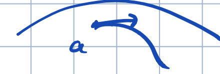
With a change of coord. $(x_1,\dots,x_n) \longmapsto (y_1(x),\dots,y_n(x))$, we get

$$df = \sum_j \frac{\partial f}{\partial y_j}\, dy_j = \sum_{i,j} \frac{\partial f}{\partial y_j}\frac{\partial y_j}{\partial x_i}\, dx_i$$

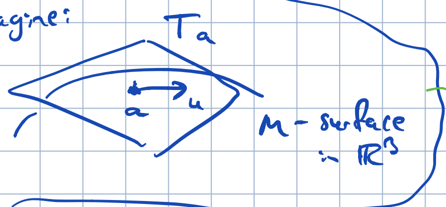
* <u>def</u> The <u>tangent space</u> T_a at $a \in M$ is the dual space to the cotangent space T_a^*.

- if $x_1,\dots,x_n$ - loc. coord. system at a and $(dx_1)_a,\dots,(dx_n)_a$ - the corresp. basis of T_a^*, the dual basis of T_a is denoted $\left(\frac{\partial}{\partial x_1}\right)_a,\dots,\left(\frac{\partial}{\partial x_n}\right)_a$

two approaches to intrinsic definition of T_a: (i) equivalence classes of curves $f: \mathbb{R} \to M$



imagine:



(ii) tangent vector $\vec{u} \rightsquigarrow$ directional derivative $f \longmapsto \vec{u}\cdot\nabla f(a)$
$\qquad \overset{\shortparallel}{Df_a(u)}$

<algebraic definition>

<u>def</u> A tangent vector at $a \in M$ is a linear map $X_a : C^\infty(M) \to \mathbb{R}$ s.t. $\boxed{X_a(fg) = f(a)X_a g + g(a) X_a f}$. (formal Leibnitz rule)

---

if $\underbrace{\xi \in T_a}_{\text{dual space to } C^\infty(M)/Z_a}$, $\quad f \longmapsto \xi((df)_a)$
$\qquad\qquad\qquad\qquad\qquad\qquad C^\infty(M) \longrightarrow \mathbb{R}$

Moreover, from (*): $d(fg)_a = f(a)(dg)_a + g(a)(df)_a$

(#) Thus, $X_a : f \longmapsto \xi((df)_a)$ is a tangent vector.

- In fact, all tangent vectors are of this form!

injective! map $T_a \to T_a^{Alg}$ (cannot have a $\xi \neq 0$ s.t. $\xi((df)_a) = 0 \ \forall f$)

$T_a \longrightarrow T_a^{Alg}$

$(\xi : T_a^* \to \mathbb{R}) \longleftarrow X_a$

$[df]_a \mapsto X_a f$

or just: $(X_a : C^\infty(M)/Z_a \to \mathbb{R}) \longleftarrow X_a$

<u>Lemma</u> Let X_a be a tangent vector at a and $f \in Z_a$. Then $X_a f = 0$.

<u>Proof.</u> Use a loc. coord. sys near a:

$$f(x) - f(a) = \int_0^1 \frac{\partial}{\partial t} f(a + t(x-a))\,dt = \sum_i (x_i - a_i)\underbrace{\int_0^1 \frac{\partial f}{\partial x_i}(a + t(x-a))\,dt}_{g_i(x)}$$

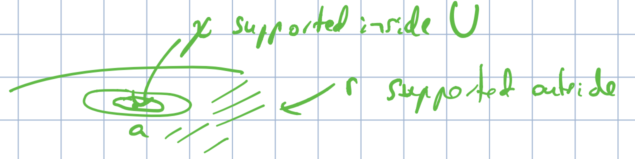
fund thm of calc

if $(df)_a = 0$, then $g_i\big|_{x=a} = 0$ and $h_i(x) = x_i - a_i$ also vanishes at $x = a$.

$$\begin{cases} f = f(a) + \sum_i g_i h_i & \text{- locally, near } a \quad \text{with } g_i, h_i \text{ vanishing at } x=a. \\ f = f(a) + \sum_i \tilde{g}_i \tilde{h}_i + \underbrace{r}_{\text{vanishes in nbhd of } a} & \text{- globally} \end{cases}$$

$\tilde{g}_i = \psi g_i$, $\tilde{h}_i = \psi h_i$

global extension by a bump fun. ψ



- $X_a(\underbrace{r \cdot \chi}_{0}) = \underbrace{\chi(a)}_{1} X_a(r) + \underbrace{r(a)}_{0} X_a(\chi) = X_a(r) \Rightarrow X_a(r) = 0$ (χ: bump near a)
- $X_a(1 \cdot 1) \overset{\text{Leibnitz}}{=} 1 \cdot X_a(1) + 1\, X_a(1) \Rightarrow X_a(1) = 0 \Rightarrow X_a(\text{any const function}) = 0$

$$\Rightarrow X_a f = \sum_i \tilde{g}_i(a) X_a(\tilde{h}_i) + \tilde{h}_i(a) X_a(\tilde{g}_i) = 0 \qquad \square$$

- if $V \subset W$ vector spaces, $\mathrm{Ann}(V) \subset W^*$ annihilator,
  then $\mathrm{Ann}(V) \cong (W/V)^*$

  set $W = C^\infty(M)$, $V = Z_a$ then

$$\begin{array}{c} (C^\infty(M)/Z_a)^* = T_a \\ \text{\rotatebox{90}{$\cong$}} \\ \mathrm{Ann}(Z_a) \subset (C^\infty(M))^* \\ \text{\rotatebox{90}{$\subseteq$}} \\ \{\text{tangent vectors}\} \\ \| \\ T_a^{alg} \end{array}$$

Lemma

$\Rightarrow T_a^{alg} \subseteq T_a$

- together with (#), it gives $T_a \cong T_a^{alg}$.

Thus, vectors in T_a are the tangent vectors

Locally; in coordinates: $X_a = \sum_{i=1}^n c_i \left(\frac{\partial}{\partial x_i}\right)_a$

then $X_a f = \sum_{i=1}^n c_i \frac{\partial f}{\partial x_i}(a) \qquad (@)$

Derivatives of smooth maps

Suppose $F: M \to N$ smooth map, $f \in C^\infty(N)$. Then $f \circ F \in C^\infty(M)$.

Rem: $C^\infty(N) \xrightarrow{F^*} C^\infty(M)$ is a homomorphism of rings.
$f \longmapsto f \circ F$

$F^* f$ – "pullback of f along F"

def The <u>derivative</u> at $a \in M$ of the smooth map $F: M \to N$ is the homomorphism of tangent spaces $DF_a : T_aM \to T_{F(a)}N$ defined by

$$DF_a(X_a)(f) = X_a(f \circ F)$$

- This is an abstract, coord-free definition. In coordinates, using (@):

$$DF_a\left(\frac{\partial}{\partial x_i}\right)_a (f) = \frac{\partial}{\partial x_i}(f \circ F)(a) = \sum_j \frac{\partial F_j}{\partial x_i}(a) \frac{\partial f}{\partial y_j}(F(a)) = \sum_j \frac{\partial F_j}{\partial x_i}(a) \left(\frac{\partial}{\partial y_j}\right)_{F(a)} f$$

i.e. $DF_a : \left(\frac{\partial}{\partial x_i}\right)_a \longmapsto \sum_j \frac{\partial F_j}{\partial x_i}(a)\left(\frac{\partial}{\partial y_j}\right)_{F(a)}$

thus, DF_a is an invariant way of defining the Jacobian matrix.

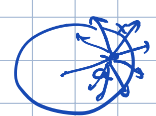
Thm Let $F: M \to N$ be a smooth map and $c \in N$ be such that for each $a \in F^{-1}(c)$, the derivative DF_a is surjective. (i.e. c is a "reg. value of F") Then $F^{-1}(c)$ is a smooth manifold of dimension $\dim M - \dim N$.

- inclusion $\iota: F^{-1}(c) \hookrightarrow M$ is a smooth map, $D\iota$ is injective and $\operatorname{im} D\iota_a = \ker DF_a$

Thus: $T_aF^{-1}(c) \cong \ker DF_a$ — helps understand tangent spaces in the case $M = \mathbb{R}^n$!

<u>Examples</u>: 1) $S^n = F^{-1}(1)$, $F: \mathbb{R}^{n+1} \to \mathbb{R}$, $x \mapsto \|x\|^2$ $\qquad DF_a(x) = \sum_i 2x_i a_i$

$\ker DF_a = \{\text{vectors orthogonal to } a \in S^n\}$



2) $O(n) = F^{-1}(I)$, $F: \mathrm{Mat}_{n\times n} \to \mathrm{Sym\,Mat}_{n\times n}$, $A \mapsto A^TA$ $\qquad DF_I(H) = H^T + H$

$\ker DF_I = \{H \in \mathrm{Mat}_{n\times n} \mid H^T = -H\} = \{\text{skew-sym. matrices}\}$.

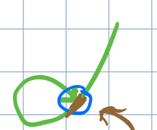
def A manifold M is an (embedded) submanifold of N if there is an inclusion map $\iota: M \to N$ s.t.

(a) ι is smooth

(b) $D\iota_x$ is injective for each $x \in M$

(c) the topology on M coincides with the induced (subspace) one from N.

to avoid a situation like



$(-1,\infty) \xrightarrow{\iota} \mathbb{R}^2$
$t \longmapsto (t^2-1,\, t(t^2-1))$

$\iota(1-\delta, 1+\delta)$ not open in induced topology!