Proposition 5.7 If $\dim V = n$ and $A : V \to V$, then the linear transformation $\Lambda^n V \to \Lambda^n V$ is given by (multiplication by) $\det A$.

determinant of the matrix of A rel. to some basis

Proof: $\Lambda^n V$ is 1-dimensional, so $\Lambda^n A : \Lambda^n V \to \Lambda^n V$ is a multiplication by a real number $\lambda(A)$. For $v_1, \dots, v_n$ a basis in V,

$$\Lambda^n A(v_1 \wedge \dots \wedge v_n) = A v_1 \wedge \dots \wedge A v_n = \lambda(A)\, v_1 \wedge \dots \wedge v_n$$

$$A v_i = \sum_j A_{ji} v_j$$

$$\left(\Rightarrow A\left(\sum_i v_i x_i\right) = \sum_{i,j} v_j \underline{A_{ji} x_i} \right) \quad x'_j \text{ - coords of transformed vector}$$

$$A v_1 \wedge \dots \wedge A v_n = \sum_{j_1 \dots j_n} A_{j_1 1} v_{j_1} \wedge \dots \wedge A_{j_n n} v_{j_n}$$

$$= \sum_{\sigma \in S_n} A_{\sigma(1) 1} v_{\sigma(1)} \wedge \dots \wedge A_{\sigma(n) n} v_{\sigma(n)}$$

permutations

$$= \underbrace{\sum_{\sigma \in S_n} \operatorname{sign} \sigma \cdot A_{\sigma(1) 1} \cdots A_{\sigma(n) n}}_{\det A} \; v_1 \wedge \dots \wedge v_n$$

$\square$

Differential forms

The bundle of p-forms. Let M - n-manifold, T_x^* - cotangent space, $\Lambda^p T_x^*$ - its exterior power

consider $\Lambda^p T^* M := \bigcup_{x \in M} \Lambda^p T_x^*$ - we will endow it with the structure of a vector bundle (and thus a manifold)

(with a natural projection π to M)

- Let (U, φ_U) a chart on M with $x_1, \dots, x_n$ coordinates.

Then for $x \in U$ the elements $dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_p}$ for $i_1 < \dots < i_p$ form a basis of $\Lambda^p T_x^*$.

The $\binom{n}{p}$ coefficients of $\alpha \in \Lambda^p T_x^*$ then give a coord. chart Ψ_U

$$\underbrace{\bigcup_{x \in U} \Lambda^p T_x^*}_{\pi^{-1}(U)} \xrightarrow{\Psi_U} \varphi_U(U) \times \mathbb{R}^{\binom{n}{p}} \underset{\text{open}}{\subset} \mathbb{R}^n \times \mathbb{R}^{\binom{n}{p}}$$

$$\sum_{i_1 < \dots < i_p} y_{i_1 \dots i_p} (dx_{i_1} \wedge \dots \wedge dx_{i_p})_x \longmapsto (\varphi_U(x) \,;\, \{y_{i_1 \dots i_p}\}_{i_1 < \dots < i_p})$$

- for $p = 1$, this is just the coord. chart we had for the cotangent bundle T^*M

$$\Psi_U\left(x, \sum_i y_i dx_i\right) = (x_1, \dots, x_n; y_1, \dots, y_n)$$ and on an overlap $U_\alpha \cap U_\beta$ we had

$$\Psi_\beta \circ \Psi_\alpha^{-1}(x_1, \dots, x_n; y_1, \dots, y_n) = \left(\tilde{x}_1, \dots, \tilde{x}_n; \sum_j \frac{\partial x_j}{\partial \tilde{x}_1} y_j, \dots, \sum_j \frac{\partial x_j}{\partial \tilde{x}_n} y_j\right)$$ there is a mistake in Hitchin here!

- For p arbitrary, we replace the matrix $A = \underbrace{\frac{\partial x_j}{\partial \tilde{x}_i}}_{Jac^{-1T}}$ by its p-th exterior power

$\Lambda^p A : \Lambda^p \mathbb{R}^n \to \Lambda^p \mathbb{R}^n$
- complicated to write in a basis, but
- invertible
- C^∞ in x

$$\sum_{j_1<\dots<j_p} y_{j_1\dots j_p} dx_{j_1} \wedge \dots \wedge dx_{j_p} = \sum_{i_1<\dots<i_p} \tilde{y}_{i_1\dots i_p} d\tilde{x}_{i_1} \wedge \dots \wedge d\tilde{x}_{i_p}$$
iff $\tilde{y}_{i_1\dots i_p} = \sum_{\sigma \in S_p} \operatorname{sign}\sigma \sum_{j_1<\dots<j_p} \frac{\partial x_{j_1}}{\partial \tilde{x}_{i_{\sigma(1)}}} \cdots \frac{\partial x_{j_p}}{\partial \tilde{x}_{i_{\sigma(p)}}} y_{j_1\dots j_p}$

$\Rightarrow \psi_\beta \circ \psi_\alpha^{-1}$ is C^∞ on overlaps $\Rightarrow \Lambda^p T^* M$ is a smooth manifold, $\dim = n + \binom{n}{p}$

- **def** The bundle of p-forms on M is the smooth mfd $\Lambda^p T^* M$ with the smooth structure as above; There is a natural projection $\pi : \Lambda^p T^* M \to M$; a section is called a <u>differential p-form</u>.

Ex: 1. A zero-form is a section of $\Lambda^0 T^* M$ which is by convention a smooth function f.

2. A 1-form is a section of the cotangent bundle. E.g. df is a 1-form.
In loc. coords: $df = \sum_j \frac{\partial f}{\partial x_j} dx_j$.

- By using a bump function, we can extend a locally-defined p-form like $dx_1 \wedge \dots \wedge dx_p$ to the whole M $\Rightarrow$ sections always exist.

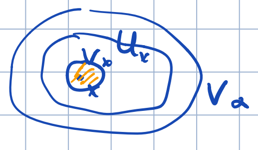
- one can represent functions / vector fields / p-forms as sums of local ones using a <u>partition of unity</u>.

- <u>Partitions of unity</u>

def A <u>partition of unity</u> on M is a collection $\{\varphi_i\}_{i \in I}$ of smooth functions s.t.
- $\varphi_i \geq 0$
- $\{\operatorname{supp} \varphi_i : i \in I\}$ is locally finite, i.e. $\forall x \in M$ $\exists U$ open nbhd which intersects only finitely many supports $\operatorname{supp} \varphi_i$.
- $\sum_i \varphi_i = 1$

Theorem (Thm 10.8) Given any open cover $\{V_\alpha\}$ of a manifold M, there exists a partition of unity $\{\varphi_i\}$ on M s.t. $\operatorname{supp} \varphi_i \subset V_{\alpha(i)}$ for some $\alpha(i)$.
(then one says that $\{\varphi_i\}$ is "subordinate" to the cover $\{V_\alpha\}$)

Proof in case M compact: $\forall x \in M$ take a coord. nbhd $U_x \subset \underset{\text{some } \alpha}{V_\alpha}$ and a bump function μ_x which is 1 in a nbhd $\underset{x}{V_x} \subset U_x$ and with $\operatorname{supp} \mu_x \subset U_x$



$\{V_x\}_{x \in M}$ is a cover of M $\underset{\text{compactness}}{\Longrightarrow}$ can find a finite subcover $\{V_{x_i}\}_{i=1\dots N}$

Set $\varphi_i = \dfrac{\mu_{x_i}}{\sum_{j=1}^{\tilde{n}} \mu_{x_j}}$ - this is a partition of unity subordinate to $\{V_\alpha\}$ $\square$

Working with differential forms

In a loc. coord. system on M, a differential form looks like

$$\alpha = \sum_{i_1<\dots<i_p} \underbrace{\alpha_{i_1\dots i_p}(x)}_{\text{smooth functions}} dx_{i_1}\wedge\dots\wedge dx_{i_p} \qquad (*)$$

If $y_1,\dots,y_n$ - different loc. coordinate system, $x = x(y)$, then write

$$dx_{i_k} = \sum_j \frac{\partial x_{i_k}}{\partial y_j} dy_j \quad \rightsquigarrow \text{ substitute into } (*) \text{ to get}$$

$$\alpha = \sum_{j_1<\dots<j_p} \tilde{\alpha}_{j_1\dots j_p}(y)\, dy_{j_1}\wedge\dots\wedge dy_{j_p}$$

Ex: Let $M = \mathbb{R}^2$, $\omega = dx_1\wedge dx_2$ - 2-form.

Change to polar coords on the open set $(x_1,x_2)\neq(0,0)$: $\leftarrow$ If we want polar coords to be a coord. chart, we should cut out more, e.g. $(x_1,x_2)\neq(-t,0)$, $t\geq 0$

$x_1 = r\cos\theta$, $x_2 = r\sin\theta$

$$\Rightarrow \begin{aligned} dx_1 &= \cos\theta\, dr - r\sin\theta\, d\theta \\ dx_2 &= \sin\theta\, dr + r\cos\theta\, d\theta \end{aligned}$$

$$\Rightarrow \omega = (\cos\theta\, dr - r\sin\theta\, d\theta)\wedge(\sin\theta\, dr + r\cos\theta\, d\theta) = r\, dr\wedge d\theta$$

Notation $\Omega^p(M) :=$ space of all p-forms on M (an ∞-dimensional vector space)

Pull-back of a differential form

Rem: for tangent vectors, we have a push forward $DF_x : T_xM \to T_{F(x)}N$ but it doesn't generally define a push forward for vector fields if F is not onto (and if F not injective, it can be ill-defined)

$F: M \to N$ a smooth map

Let $\alpha \in \Omega^p(N)$

$DF_x : T_xM \to T_{F(x)}N$

dual map $\swarrow$ so as not to confuse with pullback

$DF_x^\vee : T^*_{F(x)}N \to T^*_xM$

Λ^p

$\Lambda^p(DF_x^\vee) : \Lambda^p T^*_{F(x)}N \to \Lambda^p T^*_x M$

So: $\Lambda^p(DF_x^\vee)(\alpha_{F(x)})$ is defined for all x and is a p-form on M

def The pullback of a p-form $\alpha \in \Omega^p(N)$ by a smooth map $F: M \to N$ is the p-form $F^*\alpha \in \Omega^p(M)$ defined by $(F^*\alpha)_x := \Lambda^p(DF_x^\vee)(\alpha_{F(x)})$

Ex: ① For a 0-form $f \in C^\infty(M)$, we get the pullback of functions,

$$F^*f = f \circ F$$

② by def. of the dual map, for $\alpha \in \Omega^1(N)$

$$DF_x^\vee(\underbrace{\alpha_{F(x)}}_{\in T^*_{F(x)}N})(X_x) = \alpha_{F(x)}(DF_x(X_x))$$

take $\alpha = df$:

$$(F^*(df))(X_x) = DF_x^\vee(df)(X_x) = df_{F(x)}(DF_x(X_x)) = \underbrace{X_x(F^*f)}_{\text{by def. of DF}} = (d(F^*f))(X_x)$$

$$\Rightarrow \boxed{F^*(df) = d(F^*f)}$$

③ Let $F: \mathbb{R}^3 \to \mathbb{R}^2$, $\alpha = x\,dx\wedge dy \in \Omega^2(\mathbb{R}^2)$. Let's find $F^*\alpha$.

$$(x_1,x_2,x_3) \mapsto (x_1x_2,\, x_2+x_3) = (x,y)$$

$$\begin{aligned}
F^*\alpha &= (F^*x)\, dF^*(x) \wedge dF^*(y) \\
&= x_1x_2\, d(x_1x_2) \wedge d(x_2+x_3) \\
&= x_1x_2\,(x_1\,dx_2 + x_2\,dx_1) \wedge (dx_2 + dx_3) \\
&= x_1^2x_2\, dx_2\wedge dx_3 + x_1x_2^2\, dx_1\wedge dx_2 + x_1x_2^2\, dx_1\wedge dx_3
\end{aligned}$$

Properties of pullbacks of diff. forms

- $(F\circ G)^*\alpha = G^*(F^*\alpha)$
- $F^*(\alpha+\beta) = F^*\alpha + F^*\beta$
- $F^*(\alpha\wedge\beta) = F^*\alpha \wedge F^*\beta$
- $F^*(df) = d(F^*f)$ <saw above>