Correction to Last time: explicit transition map in $\Lambda^p T^* M$

$$\sum_{j_1<\dots<j_p} y_{j_1\dots j_p}\, dx_{j_1}\wedge\dots\wedge dx_{j_p} = \sum_{i_1<\dots<i_p} \tilde{y}_{i_1\dots i_p}\, d\tilde{x}_{i_1}\wedge\dots\wedge d\tilde{x}_{i_p}$$

iff $$\boxed{\tilde{y}_{i_1\dots i_p} = \sum_{\sigma\in S_p} \mathrm{sign}(\sigma) \sum_{j_1<\dots<j_p} \frac{\partial x_{j_1}}{\partial \tilde{x}_{i_{\sigma(1)}}} \cdots \frac{\partial x_{j_p}}{\partial \tilde{x}_{i_{\sigma(p)}}}\, y_{j_1\dots j_p}}$$

Working with differential forms

In a loc. coord. system on M, a differential form looks like

$$\alpha = \sum_{i_1<\dots<i_p} \underbrace{\alpha_{i_1\dots i_p}(x)}_{\text{smooth functions}} dx_{i_1}\wedge\dots\wedge dx_{i_p} \qquad (*)$$

If $y_1,\dots,y_n$ - different loc. coordinate system, $x=x(y)$, then write

$$dx_{i_k} = \sum_j \frac{\partial x_{i_k}}{\partial y_j} dy_j \quad \rightsquigarrow \text{ substitute into } (*) \text{ to get}$$

$$\alpha = \sum_{j_1<\dots<j_p} \tilde{\alpha}_{j_1\dots j_p}(y)\, dy_{j_1}\wedge\dots\wedge dy_{j_p}$$

Ex: Let $M=\mathbb{R}^2$, $\omega = dx_1\wedge dx_2$ - 2-form.

Change to polar coords on the open set $(x_1,x_2)\neq(0,0)$: ← If we want polar coords to be a coord. chart, we should cut out more, e.g. $(x_1,x_2)\neq(-t,0)$, $t\geq 0$

$x_1 = r\cos\theta$, $x_2 = r\sin\theta$

$$\Rightarrow \quad dx_1 = \cos\theta\, dr - r\sin\theta\, d\theta$$
$$dx_2 = \sin\theta\, dr + r\cos\theta\, d\theta$$

$$\Rightarrow \quad \omega = (\cos\theta\, dr - r\sin\theta\, d\theta)\wedge(\sin\theta\, dr + r\cos\theta\, d\theta) = r\, dr\wedge d\theta$$

Notation: $\Omega^p(M) :=$ space of all p-forms on M (an ∞-dimensional vector space)

Pull-back of a differential form

$F: M\to N$ a smooth map

Let $\alpha\in\Omega^p(N)$

$DF_x: T_xM\to T_{F(x)}N$

dual map

$DF_x^\vee: T^*_{F(x)}N \to T^*_xM$ ← so as not to confuse with pullback

Λ^p

$\Lambda^p(DF_x^\vee): \Lambda^p T^*_{F(x)}N \to \Lambda^p T^*_xM$

So: $\Lambda^p(DF_x^\vee)(\alpha_{F(x)})$ is defined for all x and is a p-form on M

Rem: for tangent vectors, we have a push forward $DF_x: T_xM\to T_{F(x)}N$ but it doesn't generally define a push forward for vector fields if F is not onto (and if F not injective, it can be ill-defined)

def The pullback of a p-form $\alpha \in \Omega^p(N)$ by a smooth map $F: M \to N$ is the p-form $F^*\alpha \in \Omega^p(M)$ defined by $(F^*\alpha)_x := \Lambda^p(DF_x^\vee)(\alpha_{F(x)})$

Ex: ① For a 0-form $f \in C^\infty(M)$, we get the pullback of functions,

$$F^*f = f \circ F$$

② by def. of the dual map, for $\alpha \in \Omega^1(N)$

$$DF_x^\vee(\underbrace{\alpha_{F(x)}}_{\in T^*_{F(x)}N})(X_x) = \alpha_{F(x)}(DF_x(X_x))$$

take $\alpha = df$:

$$(F^*(df))(X_x) = DF_x^\vee(df)(X_x) = df_{F(x)}(DF_x(X_x))$$

$$= X_x(F^*f) \quad = (d(F^*f))(X_x)$$

by def. of DF

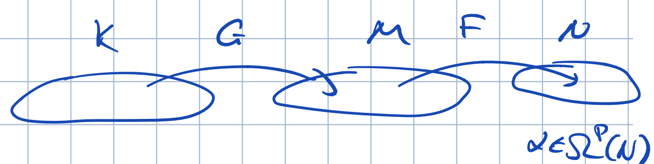
$$\Rightarrow \boxed{F^*(df) = d(F^*f)}$$

③ Let $F: \mathbb{R}^3 \to \mathbb{R}^2$, $\alpha = x\,dx \wedge dy \in \Omega^2(\mathbb{R}^2)$. Let's find $F^*\alpha$.

$(x_1, x_2, x_3) \mapsto (x_1x_2, x_2 + x_3) = (x, y)$

$$F^*\alpha = (F^*x)\, dF^*(x) \wedge dF^*(y)$$
$$= x_1x_2\, d(x_1x_2) \wedge d(x_2 + x_3)$$
$$= x_1x_2\,(x_1dx_2 + x_2dx_1) \wedge (dx_2 + dx_3)$$
$$= x_1^2x_2\, dx_2 \wedge dx_3 + x_1x_2^2\, dx_1 \wedge dx_2 + x_1x_2^2\, dx_1 \wedge dx_3$$

Properties of pullbacks of diff. forms



- $(F \circ G)^*\alpha = G^*(F^*\alpha)$
- $F^*(\alpha + \beta) = F^*\alpha + F^*\beta$
- $F^*(\alpha \wedge \beta) = F^*\alpha \wedge F^*\beta$

in homework

- $F^*(df) = d(F^*f)$ <saw above>

Exterior derivative

Theorem# On any smooth manifold M there is a natural linear map

$d: \Omega^p(M) \to \Omega^{p+1}(M)$ - the "exterior derivative" - such that

1. if $f \in \Omega^0(M)$ then $df \in \Omega^1(M)$ is the derivative of f
2. $d^2 = 0$
3. $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ if $\alpha \in \Omega^p(M)$.