LAST TIME: exterior derivative $d: \Omega^p(M) \to \Omega^{p+1}(M)$

- $d^2 = 0$
- $d(\underbrace{\alpha}_{p\text{-form}} \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$
- $\underbrace{df}_{\text{as a 0-form}} = \underbrace{df}_{\text{as a function}}$

locally: $\alpha = \sum_{i_1 < \dots < i_p} a_{i_1 \dots i_p}(x)\, dx_{i_1} \wedge \dots \wedge dx_{i_p} \Rightarrow d\alpha = \sum_{i_1 < \dots < i_p} da_{i_1 \dots i_p} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p}$

- if $F: M \to N$, $\alpha \in \Omega^p(N)$, then
$$d(F^*\alpha) = F^*(d\alpha)$$

Lie derivative of a differential form

def Let X be a vector field on a manifold M and $\alpha \in \Omega^p(M)$ a p-form. Lie derivative of α along X is defined as

$$\mathcal{L}_X \alpha := \frac{d}{dt}\Big|_{t=0} \varphi_t^* \alpha$$

where $\varphi_t : \underset{\mathbb{R}\times M}{\overset{\cap}{U}} \to M$ - (local) flow on M generated by X.

Proposition: Given a vector field X on M, there is a linear map

$\iota_X : \Omega^p(M) \to \Omega^{p-1}(M)$ (the inner product, or contraction with X, or substitution of X)

such that

(i) $\iota_X\, df = X(f)$

(ii) $\iota_X(\alpha \wedge \beta) = \iota_X \alpha \wedge \beta + (-1)^p \alpha \wedge \iota_X \beta$ if $\alpha \in \Omega^p$

$\overset{p=0}{\Rightarrow} \iota_X(f\alpha) = f \cdot \iota_X \alpha$

$\Rightarrow \iota_X(df_1 \wedge \dots \wedge df_p) = X(f_1)\, df_2 \wedge \dots \wedge df_p - df_1\, X(f_2) \wedge \dots \wedge df_p \dots + (-1)^{p-1} df_1 \wedge \dots \wedge df_{p-1} X(f_p)$

Ex: if $X = \sum_i a_i \frac{\partial}{\partial x_i}$, $\alpha = f\, dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_p}$ then

$$\iota_X \alpha \overset{(i),(ii)}{=} f a_{i_1} dx_{i_2} \wedge \dots \wedge dx_{i_p} - f a_{i_2} dx_{i_1} \wedge dx_{i_3} \wedge \dots \wedge dx_{i_p} + \dots \qquad (\#)$$

$\Rightarrow \iota_X(\iota_X \alpha) = f a_{i_1} a_{i_2} dx_{i_3} \wedge \dots \wedge dx_{i_p} - f a_{i_2} a_{i_1} dx_{i_3} \wedge \dots \wedge dx_{i_p} + \dots = 0$

Ex $\alpha = dx \wedge dy$ $X = x\frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ $\Rightarrow$ $\iota_X \alpha = x\,dy - y\,dx$.

Proof: $\Lambda^p V = (\text{alternating } p\text{-multilinear forms } \mu : \underbrace{V \times \dots \times V}_{p} \to \mathbb{R})^* = \mathrm{Alt}^p(V)^*$

If $\mu : \underbrace{V\times \dots \times V}_{p-1} \to \mathbb{R}$ alt. multilin. form and $\xi \in V^*$, then

$$\underbrace{(\xi\,\mu)}_{\in \mathrm{Alt}^p(V)}(v_1, \dots, v_p) = \xi(v_1)\mu(v_2, \dots, v_p) - \xi(v_2)\mu(v_1, v_3, \dots, v_p) + \dots \quad (\#\#)$$

– an alternating p-multilin. form on V

If $\alpha \in \Lambda^p V$, define $(\iota_\xi \alpha)(\mu) := \alpha(\xi\mu)$.

Taking $V = T_x^* M$, $\xi = X_x \in V^* = T_x M$, we get the interior product;

$(\#\#) \Rightarrow (\#)$ $\rightsquigarrow$ can compute the interior product.

$\square$

Alternative (equivalent) definition of $\iota_X \alpha$.

already explained

$\alpha \in \Omega^p(M)$ can be seen [by a HW problem] as a map

$\mathfrak{X}(M)$ = space of vector fields on M.

$$\alpha : \mathfrak{X}(M) \times \dots \times \mathfrak{X}(M) \longrightarrow C^\infty(M)$$
$$(X_1, \dots, X_p) \longmapsto \alpha(X_1, \dots, X_p)$$

which is
- skew-symmetric
- $C^\infty(M)$-linear in each argument

Then: $$\boxed{(\iota_X \alpha)(X_1, \dots, X_{p-1}) := \alpha(X, X_1, \dots, X_{p-1})}$$

– a $(p-1)$-form.

check: $p=1$ $\iota_X df = df(X) = X(f)$

from this construction of ι_X, it is obvious that

$$\boxed{\iota_X(\iota_X \alpha) = 0}$$

"$\alpha(X, X, \dots)$"

- check Leibnitz property:

$$(\iota_X(\underset{\text{1-form}}{\alpha} \wedge \underset{q\text{-form}}{\beta}))(X_1 \dots X_q) = (\alpha\wedge\beta)(X, X_1, \dots, X_q) = \alpha(X)\beta(X_1 \dots X_q) - \alpha(X_1)\beta(X, X_2, \dots, X_q) + \alpha(X_2)\beta(X, X_1, X_3 \dots X_q) - \dots$$

$$(\underbrace{\iota_X\alpha}_{\alpha(X)} \wedge \beta - \alpha \wedge \iota_X\beta)(X_1 \dots X_q) = \alpha(X)\beta(X_1 \dots X_q) - \alpha(X_1)\beta(X, X_2 \dots X_q) + \alpha(X_2)\beta(X, X_1, X_3, \dots, X_q) - \dots$$

already explained

$\Lambda^p T_x^*$
$\|$
$\mathrm{Alt}(T_x^* \times \dots \times T_x^* \to \mathbb{R})^*$
$\simeq \mathrm{Alt}(T_x \times \dots \times T_x \to \mathbb{R})$

$$(\alpha_1 \wedge \dots \wedge \alpha_p)(X_1, \dots, X_p) := \sum_{\sigma \in S_p} \operatorname{sign}\sigma \; \alpha_1(X_{\sigma(1)}) \cdots \alpha_p(X_{\sigma(p)})$$

1-forms

$$(\underset{\in\Omega^p}{\alpha} \wedge \underset{\in\Omega^q}{\beta})(X_1 \dots X_{p+q}) = \sum_{\sigma \in \mathrm{Shuffles}_{p,q}} \operatorname{sign}(\sigma)\, \alpha(X_{\sigma(1)}, \dots, X_{\sigma(p)})\, \beta(X_{\sigma(p+1)}, \dots, X_{\sigma(p+q)})$$

$\Leftarrow$ permutations in S_{p+q} s.t. $\sigma(1) < \dots < \sigma(p)$, $\sigma(p+1) < \dots < \sigma(p+q)$

(anti-symmetrically)

So, at a point, a p-form eats p tangent vectors and spits out a number!

Proposition For α a p-form on M and X a vector field, the Lie derivative is: $\mathcal{L}_X \alpha = d(\iota_X \alpha) + \iota_X(d\alpha)$

Proof Denote rhs $= R_X(\alpha) = d\,\iota_X \alpha + \iota_X\, d\alpha$. • R_X maps p-forms to p-forms

We have: • $R_X(d\alpha) = d\,\iota_X d\alpha + \cancel{\iota_X dd\alpha} = d\,R_X(\alpha)$ ← R_X commutes with d.

• $R_X(\alpha \wedge \beta) = R_X\alpha \wedge \beta + \alpha \wedge R_X \beta$ — since $\iota_X(\alpha\wedge\beta) = \iota_X\alpha \wedge \beta + (-1)^p \alpha \wedge \iota_X \beta$
$d(\alpha\wedge\beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$

On the other hand, $\varphi_t^*(d\alpha) = d\,\varphi_t^*(\alpha) \underset{\frac{d}{dt}\big|_{t=0}}{\Longrightarrow} \mathcal{L}_X d\alpha = d\,\mathcal{L}_X \alpha$

$\varphi_t^*(\alpha\wedge\beta) = \varphi_t^*(\alpha) \wedge \varphi_t^*(\beta) \underset{\frac{d}{dt}\big|_{t=0}}{\Longrightarrow} \mathcal{L}_X(\alpha\wedge\beta) = \mathcal{L}_X\alpha \wedge \beta + \alpha \wedge \mathcal{L}_X \beta$

So: $\mathcal{L}_X$ and R_X - preserve degree, commute with d, satisfy (same) Leibnitz identity

$\Rightarrow$ for $\alpha = \sum\limits_{i_1 < \dots < i_p} \alpha_{i_1 \dots i_p}(x)\, dx_{i_1} \wedge \dots \wedge dx_{i_p}$ $\mathcal{L}_X$ and R_X agree if they agree on functions.

$R_X f = \iota_X df = X(f) = \frac{d}{dt}\Big|_{t=0} f(\varphi_t) = \mathcal{L}_X(f)$ $\square$

by def. of a flow of X

de Rham cohomology

* Motivation from Calc 3: if $U \subset \mathbb{R}^3$ simply-connected, then $\operatorname{curl} \vec{v} = 0$ implies $\vec{v} = \operatorname{grad} f$ for some f

In the language of diff. forms:

$\alpha = df \Rightarrow d\alpha = 0$. But if U is simply-connected, it is an equivalence!

$f \in \Omega^0(U)$, $\alpha \in \Omega^1(U)$

Def The p-th de Rham cohomology group of a manifold M is the quotient vector space:

$$H^p(M) = \frac{\ker d: \Omega^p(M) \to \Omega^{p+1}(M)}{\operatorname{im} d: \Omega^{p-1}(M) \to \Omega^p(M)}$$

Rem • "group" here = vector space$/\mathbb{R}$

• $H^0(M) = \{\text{locally constant functions}\}$ ($df = 0$)

If M connected, $H^0(M) \simeq \mathbb{R}$

if M has N connected components, $H^0(M) \simeq \mathbb{R}^N$

• **Claim**: for M compact, $H^p(M)$ is a finite-dimensional vector space. [← without proof]

Def A form $\alpha \in \Omega^p(M)$ is "closed" if $d\alpha = 0$

Form α is "exact" if $\alpha = d\beta$ for some $\beta \in \Omega^{p-1}(M)$

So, $H^p = \{\text{closed } p\text{-forms}\} / \{\text{exact } p\text{-forms}\}$

closed p-form $\alpha \longmapsto [\alpha] \in H^p$

$[\alpha'] = [\alpha]$ iff $\alpha' - \alpha = d\beta$ for some β.

Proposition de Rham cohomology groups of an n-dimensional manifold M have the following properties:

① $H^p(M) = 0$, $p > n$

② for $a \in H^p(M)$, $b \in H^q(M)$, there is a bilinear product $ab \in H^{p+q}(M)$ satisfying $ab = (-1)^{pq} ba$

③ If $F: M \to N$ a smooth map, it defines a natural linear map $F^*: H^p(N) \to H^p(M)$ which commutes with the product.

Proof ① - clear since $\Lambda^{>n} T^* = 0$. ✓

② if $a = [\alpha]$, $b = [\beta]$, set $ab = [\alpha \wedge \beta]$
($\alpha \in \Omega^p_{closed}$, $\beta \in \Omega^q_{closed}$)

Need to check that $\alpha \wedge \beta$ is closed: $d(\alpha\wedge\beta) = \underbrace{d\alpha}_{0} \wedge \beta + (-1)^p \alpha \wedge \underbrace{d\beta}_{0} = 0$

$\Rightarrow \alpha\wedge\beta$ defines a cohomology class.

- need to check that it is well-defined: suppose we choose another representative for α, $\alpha' = \alpha + d\gamma$. Then $\alpha'\wedge\beta = (\alpha+d\gamma)\wedge\beta = \alpha\wedge\beta + \underbrace{d\gamma\wedge\beta}_{d(\gamma\wedge\beta)}$ - in the same cohomology class as $\alpha\wedge\beta$

changing $\beta \mapsto \beta' = \beta + d\delta$ - works similarly.

✓

(3) $\underset{\Omega^p_{cl}(M)}{F^*[\alpha]} := [\underbrace{F^*\alpha}_{\Omega^p(M)}]$

- $F^*\alpha$ closed $\Rightarrow [F^*\alpha]$ defined
- $\alpha \mapsto \underset{=\alpha'}{\alpha + d\gamma} \Rightarrow [\underbrace{F^*\alpha'}_{F^*\alpha + d F^*\gamma}] = [F^*\alpha]$

So, the operation is well-defined

$$F^*([\alpha][\beta]) = F^*[\alpha\wedge\beta] = [F^*(\alpha\wedge\beta)] = [F^*\alpha\wedge F^*\beta] = [F^*\alpha][F^*\beta]$$

- So, F^* respects the product

✓

□

We say that $F: M\times[a,b]\to N$ is smooth if it is a restriction of a smooth $\tilde{F}: M\times(a-\varepsilon, b+\varepsilon)\to N$ for some $\varepsilon>0$

Theorem Let $F: M\times[0,1]\to N$ be a smooth map. Set $F_t(x) = F(x,t)$ and consider the induced map on de Rham cohomology $F_t^*: H^p(N)\to H^p(M)$.

Then: $F_1^* = F_0^*$.