LAST TIME:

- $\mathcal{L}_X \alpha := \frac{d}{dt}\Big|_{t=0} \varphi_t^* \alpha$ - Lie derivative of $\alpha \in \Omega^p(M)$ along a vector field X

here $\varphi_t : \underset{M\times\mathbb{R}}{U} \to M$ - flow of X

- inner product $\iota_X : \Omega^p(M) \to \Omega^{p-1}(M)$

$$(\iota_X \alpha)(X_1, \dots, X_{p-1}) := \alpha(X, X_1, \dots, X_{p-1})$$

satisfies:
- $\iota_X(df) = X(f)$
- $\iota_X(\underset{p\text{-form}}{\alpha} \wedge \beta) = \iota_X \alpha \wedge \beta + (-1)^p \alpha \wedge \iota_X \beta$
- $\iota_X(\iota_X \alpha) = 0$

- Cartan's formula: $\mathcal{L}_X \alpha = d(\iota_X \alpha) + \iota_X(d\alpha)$

de Rham cohomology

* Motivation from Calc 3: if $U \subset \mathbb{R}^3$ simply-connected, then $\operatorname{curl} \vec{v} = 0$ implies $\vec{v} = \operatorname{grad} f$ for some f

In the language of diff. forms:

$\alpha = df \Rightarrow d\alpha = 0$. But if U is simply-connected, it is an equivalence!

$f \in \Omega^0(U), \alpha \in \Omega^1(U)$

Def The p-th de Rham cohomology group of a manifold M is the quotient vector space:

$$H^p(M) = \frac{\ker d : \Omega^p(M) \to \Omega^{p+1}(M)}{\operatorname{im} d : \Omega^{p-1}(M) \to \Omega^p(M)}$$

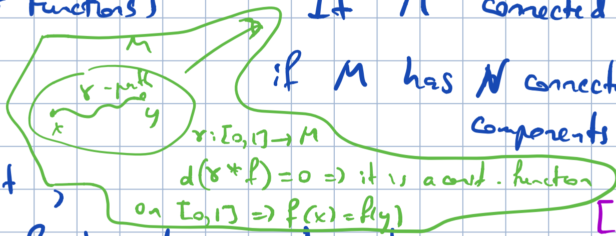
Rem • "group" here = vector space$/\mathbb{R}$

[Fact: for manifolds (even topological ones), connected = path connected]

• $H^0(M) = \{$ locally constant functions $\}$ $df = 0$

If M connected, $H^0(M) \simeq \mathbb{R}$

if M has N connected components, $H^0(M) \simeq \mathbb{R}^N$



• **Claim**: for M compact, $H^p(M)$ is a finite-dimensional vector space. [← without proof]

Def A form $\alpha \in \Omega^p(M)$ is "closed" if $d\alpha = 0$

Form α is "exact" if $\alpha = d\beta$ for some $\beta \in \Omega^{p-1}(M)$

So, $H^p = \{\text{closed } p\text{-forms}\} / \{\text{exact } p\text{-forms}\}$

closed p-form $\alpha \longmapsto [\alpha] \in H^p$

$[\alpha'] = [\alpha]$ iff $\alpha' - \alpha = d\beta$ for some β.

Proposition de Rham cohomology groups of an n-dimensional manifold M have the following properties:

1. $H^p(M) = 0$, $p > n$
2. for $a \in H^p(M)$, $b \in H^q(M)$, there is a bilinear product $ab \in H^{p+q}(M)$ satisfying $ab = (-1)^{pq} ba$
3. If $F: M \to N$ a smooth map, it defines a natural linear map $F^*: H^p(N) \to H^p(M)$ which commutes with the product.

Proof ① - clear since $\Lambda^{>n} T^* = 0$. ✓

② if $a = [\alpha]$, $b = [\beta]$, set $ab = [\alpha \wedge \beta]$
$\alpha \in \Omega^p_{\text{closed}}$, $\beta \in \Omega^q_{\text{closed}}$

Need to check that $\alpha \wedge \beta$ is closed: $d(\alpha \wedge \beta) = \underbrace{d\alpha}_{0} \wedge \beta + (-1)^p \alpha \wedge \underbrace{d\beta}_{0} = 0$

$\Rightarrow \alpha\wedge\beta$ defines a cohomology class.

- need to check that it is well-defined: suppose we choose another representative for a, $\alpha' = \alpha + d\gamma$. Then $\alpha'\wedge\beta = (\alpha+d\gamma)\wedge\beta = \alpha\wedge\beta + \underbrace{d\gamma\wedge\beta}_{d(\gamma\wedge\beta)}$ - in the same cohomology class as $\alpha\wedge\beta$

changing $\beta \mapsto \beta' = \beta + d\delta$ - works similarly.

✓

(3) $\underset{\Omega^p_{cl}(N)}{F^*[\alpha]} := [\underbrace{F^*\alpha}_{\Omega^p(M)}]$

- $F^*\alpha$ closed $\Rightarrow [F^*\alpha]$ defined
- $\alpha \mapsto \underbrace{\alpha + d\gamma}_{=\alpha'} \Rightarrow [\underbrace{F^*\alpha'}_{F^*\alpha + dF^*\gamma}] = [F^*\alpha]$

So, the operation is well-defined

$F^*([\alpha][\beta]) = F^*[\alpha\wedge\beta] = [F^*(\alpha\wedge\beta)] = [F^*\alpha\wedge F^*\beta] = [F^*\alpha][F^*\beta]$

- so, F^* respects the product

✓

□

We say that $F: M\times[a,b]\to N$ is smooth if it is a restriction of a smooth $\tilde F: M\times(a-\varepsilon, b+\varepsilon)\to N$ for some $\varepsilon > 0$

< or, more generally: $\tilde F: U \to N$, $U \supset M\times[a,b]$, $U \subset M\times\mathbb{R}$ open >

Theorem Let $F: M\times[0,1]\to N$ be a smooth map. Set $F_t(x) = F(x,t)$ and consider the induced map on de Rham cohomology $F_t^*: H^p(N)\to H^p(M)$.

Then: $F_1^* = F_0^*$.

Proof Let $a = [\alpha] \in H^p(N)$ ← closed form on N

$F^*\alpha \in \Omega^p(M\times[0,1])$ in fact - on $M\times(-\varepsilon, 1+\varepsilon)$

(#) $F^*\alpha = \beta + dt\wedge\gamma$ ← splitting

β: p-form on M depending on t; γ: $(p-1)$-form on M depending on t

splitting (#) is clear in a coord. system.

more invariantly: $\beta = F_t^*\alpha$.

To get γ: let $\varphi_t: (x,s)\mapsto(x,s+t)$ be the flow on $M\times(a,b)$ - it generates a vector field $X = \frac{\partial}{\partial t}$

then: $\gamma = \iota_X F^*\alpha$

α closed $\Rightarrow$ $F^*\alpha$ closed $\Rightarrow$ $0 = d(\beta + dt\wedge\gamma)$

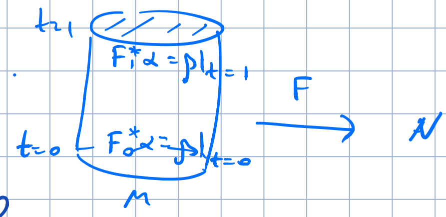
$$= d_M\beta + dt \wedge \frac{\partial\beta}{\partial t} - dt\wedge d_M\gamma$$

d_M = exterior derivative in the variables of M.

$\Rightarrow \dfrac{\partial\beta}{\partial t} = d_M\gamma$ $\quad\Rightarrow F_1^*\alpha - F_0^*\alpha$

"$\dfrac{\partial}{\partial t}F_t^*\alpha$"

$$= \int_0^1 dt\, \underbrace{\frac{\partial}{\partial t} F_t^*\alpha}_{\frac{\partial\beta}{\partial t} = d_M\gamma} = d_M \int_0^1 dt\, \gamma$$



$\Rightarrow$ closed forms $F_1^*\alpha$, $F_0^*\alpha$ differ by an exact form

$\Rightarrow$ $F_1^*a = F_0^*a$

$\boxed{}$

<u>Corollary</u>: The de Rham cohomology groups of $M = \mathbb{R}^n$ are zero for $p > 0$.

<u>Proof</u>: set $F: \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$

$(x, t) \mapsto tx$

$F_1 = id$

F_0 maps $\mathbb{R}^n$ to $0 \in \mathbb{R}^n$.

- constant map

$\underset{=id}{F_1^*}: H^p(\mathbb{R}^n) \to H^p(\mathbb{R}^n)$

$\Rightarrow$ DF_0 vanishes $\Rightarrow$

for any $\alpha \in \Omega^{>0}(\mathbb{R}^n)$, $\boxed{F_0^*\alpha = 0}$

$\Rightarrow$ $\underset{\text{zero map}}{F_0^*: H^p(\mathbb{R}^n) \to H^p(\mathbb{R}^n)}$, $p > 0$

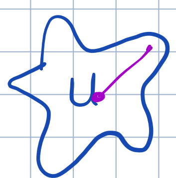
by Proposition, $F_1^* = F_0^* \Rightarrow H^p(\mathbb{R}^n) = 0$, $p > 0$ $\boxed{}$

- $\mathbb{R}^n$ is connected $\Rightarrow$ $H^0(\mathbb{R}^n) = \{$locally constant functions on $\mathbb{R}^n\} \cong \mathbb{R}$

- By a similar argument:

- <u>Poincaré lemma</u>: if $U \underset{\text{open}}{\subset} \mathbb{R}^n$ a "star-shaped region"

$\forall x \in U$, $t \in [0,1]$, $tx \in U$

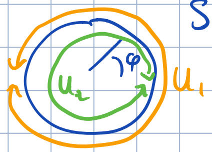


then $H^p(U) = \begin{cases} \mathbb{R} & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases}$

- $H^p(M \times \mathbb{R}^n) \cong H^p(M)$ for any manifold M

$F_t(a, x) = (a, tx)$ $\quad\sim$ deformation retraction of $M \times \mathbb{R}^n$ onto M

* Example: $H^1(S^1)$



$S^1 = \{ e^{i\varphi} \in \mathbb{C} \}$

parametrize by the angle $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$

$\mu = d\varphi$ is a nowhere-vanishing 1-form on S^1.

μ defined by this f-la in $U_1 = S^1 \setminus \{-1\}$ where $\varphi \in (-\pi, \pi)$ and $U_2 = S^1 \setminus \{+1\}$ where $\varphi \in (0, 2\pi)$

$d\mu = 0$ obviously (e.g. for degree reason)
$\Rightarrow$ μ closed

assume $\mu = \underbrace{df}_{}$ global function on S^1. S^1 compact $\Rightarrow$ f must have a min and a max $\Rightarrow$ df must vanish somewhere but μ is nonvanishing! $\Rightarrow$ μ is not exact.

So: $H^1(S^1) \neq 0$

and contains the nonzero class $[\mu]$

Let $\alpha \in \Omega^1(S^1)$ any form
$\alpha = \underbrace{g(\varphi)}_{\text{periodic function of } \varphi} d\varphi$

want $\alpha = dh$ $\Rightarrow$ $g(\varphi) = h'(\varphi)$

solution $h(\varphi) = \int_0^{\varphi} ds\, g(s)$ $(+C)$

this solution is periodic if $\int_0^{2\pi} ds\, g(s) = 0$

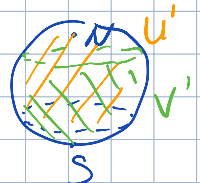
generally: $g(\varphi) = \underbrace{\frac{1}{2\pi}\int_0^{2\pi} ds\, g(s)}_{g_0 \text{ - mean value of } g \text{ on } S^1} + \tilde{g}(\varphi)$ with $\tilde{g}(\varphi) = h'(\varphi)$, $\tilde{g}(\varphi) = g(\varphi) - g_0$, $h(\varphi) = \int_0^{\varphi} ds\, \tilde{g}(s)$

So: $\alpha = g_0 \underbrace{\mu}_{d\varphi} + \underbrace{dh}_{\text{exact form}}$ $\Rightarrow$ $H^1(S^1) = \mathrm{Span}[\mu] = \mathbb{R}$. ✓

Theorem: For $n > 0$, $H^p(S^n) \cong \begin{cases} \mathbb{R} & \text{if } p = 0 \text{ or } n \\ 0 & \text{otherwise} \end{cases}$

Proof. Let $n > 1$ (case $n = 1$ was discussed above). Let $1 < p < n$.

Let $\alpha \in \Omega^p_{closed}(S^n)$



$U = S^n \setminus S \xrightarrow{\varphi_U} \mathbb{R}^n$
$V = S^n \setminus N \xrightarrow{\varphi_V} \mathbb{R}^n$
stereographic charts on S^n

$\alpha|_U = du$ ← since $H^p(\mathbb{R}^n) = 0$
$\alpha|_V = dv$ for some $u \in \Omega^{p-1}(U)$, $v \in \Omega^{p-1}(V)$

On $U \cap V$: $0 = \alpha|_{U\cap V} - \alpha|_{U \cap V} = du - dv = d(u - v)$

$\Rightarrow \quad u-v \in \Omega^{p-1}_{closed}(\underbrace{U\cap V}_{\underset{diffeo}{\cong} \mathbb{R}\times S^{n-1}})$

$$H^{p-1}(\mathbb{R}\times S^{n-1}) = H^{p-1}(S^{n-1}) \underset{\text{by induction}}{=} 0 \quad \text{for } \underline{\underline{1<p<n}}$$

$\Rightarrow u-v = dw \ , \ w \in \Omega^{p-2}(U\cap V)$

Let $U' = \varphi_U^{-1}(B_2(0))$, $V' = \varphi_V^{-1}(B_2(0))$

Let ψ = bump function with support on $U\cap V$ s.t. $\psi = 1$ on $U'\cap V'$

$\psi \cdot w$ - global $(p-2)$-form on S^n (extended by zero)

define $\beta = \begin{cases} u & \text{on } U' \\ v + d(\psi w) & \text{on } V' \end{cases}$ - global $(p-1)$-form (restrictions to $U'\cap V'$ agree)

then $d\beta = \alpha \Rightarrow \alpha$ is exact! $\Rightarrow H^p(S^n) = 0$ for $1<p<n$

- If $p=1$, $u-v \in \Omega^0_{closed}(U\cap V)$
  $= c$ - a constant function (using that $U\cap V$ is connected, for $n>1$)

  $\Rightarrow \beta = \begin{cases} u & \text{on } U \\ v+c & \text{on } V \end{cases}$ - global function, $d\beta = \alpha$ (agree on overlap)

- If $p=n$, $u-v$ defines a class in $H^{p-1}(U\cap V) \cong H^{p-1}(S^{n-1}) \underset{\text{induction}}{\cong} \mathbb{R}$

  Let $H^{p-1}(S^{n-1}) = \mathrm{Span}[\omega]$, $\omega \in \Omega^{p-1}_{closed}(S^{n-1})$

$$U\cap V \xrightarrow[\cong]{\Phi} S^{n-1}\times\mathbb{R} \xrightarrow{\pi} S^{n-1}$$

So: $u - v = \lambda\Phi^*\omega + dw$, some $\lambda \in \mathbb{R}$. If $\lambda = 0$, then we do as above and find a global $(p-1)$ form β s.t. $\alpha = d\beta$.

- λ is linear in α and independent of the choice of u, v (shifting them by an exact term can be absorbed into w)

$\Rightarrow \dim H^n(S^n) \leq 1$

- Need to find α with nonzero λ. Set $\alpha = \Phi^*(\psi(t)dt\wedge\omega)$ $\in \Omega^n(S^n)$, ψ - bump function on $\mathbb{R}$, extended by zero outside supp $\Phi^*\psi$

$u = \Phi^*\left(\left(\int_{-\infty}^{t}\psi(s)ds\right)\omega\right) \in \Omega^{p-1}(U)$ s.th. $du = \alpha$ (extension by zero from $U\cap V$ to U)

$v = \Phi^*\left(-\int_t^{\infty}\psi(s)ds\right)\omega\right) \in \Omega^{p-1}(V)$ $\qquad u - v = \underbrace{\left(\int_{-\infty}^{\infty}\psi(s)ds\right)}_{\lambda>0}\Phi^*\omega$ on $U\cap V$

$\square$