LAST TIME

$$TM = \bigcup_{x \in M} T_x M$$

tangent bundle

chart on M — "local trivialization"

$$U \times \mathbb{R}^n \xrightarrow{\psi_U} \bigcup_{x \in U} T_x M =: V_U \; ; \; \left(V_U, \Phi_U = (\varphi_U; id) \circ \psi_U^{-1}\right)$$

$$(x, y_1, \dots, y_n) \longmapsto \sum_i y_i \left(\frac{\partial}{\partial x_i}\right)_x$$

– chart on TM

transition map: $\Phi_\beta \Phi_\alpha^{-1} : (x_1, \dots, x_n; y_1, \dots, y_n) \longmapsto$

$$\longmapsto \left(\tilde{x}_1, \dots, \tilde{x}_n, \sum_i \frac{\partial \tilde{x}_1}{\partial x_i} y_i, \dots, \sum_i \frac{\partial \tilde{x}_n}{\partial x_i} y_i\right)$$

$$T^*M = \bigcup_{x \in M} T_x^* M$$

cotangent bundle

$$U \times \mathbb{R}^n \xrightarrow{\lambda_U} \bigcup_{x \in U} T_x^* M =: W_U \; ; \; \left(W_U, \Psi_U = (\varphi_U; id) \circ \lambda_U^{-1}\right)$$

$$(x; z_1, \dots, z_n) \longmapsto \sum_i z_i (dx_i)_x$$

– chart on T^*M

transition map: $\Psi_\beta \Psi_\alpha^{-1} : (x_1, \dots, x_n; z_1, \dots, z_n) \longmapsto$

$$\longmapsto \left(\tilde{x}_1, \dots, \tilde{x}_n; \sum_i \frac{\partial x_i}{\partial \tilde{x}_1} z_i, \dots, \sum_i \frac{\partial x_i}{\partial \tilde{x}_n} z_i\right)$$

The derivative of $f \in C^\infty(M)$ is a map $df: M \to T^*M$
satisfying $p \circ df = id_M$. (but not every section of T^*M is a derivative!)

TM and T^*M are examples of <u>vector bundles</u>.

<u>def</u> A real <u>vector bundle</u> of rank m on a manifold M is a manifold E with a smooth projection map $p: E \to M$ s.t.

- each fiber $p^{-1}(x)$ has the structure of an m-dimensional real vector space.
- M is covered by open sets $\{U_\alpha\}$ (trivializing neighborhoods) equipped with diffeomorphisms $\psi_\alpha: p^{-1}(U_\alpha) \xrightarrow{\simeq} U_\alpha \times \mathbb{R}^m$ ("local trivialization")
  s.t. $proj_1 \circ \psi_\alpha = p$ ($\Leftrightarrow \psi_\alpha : p^{-1}(x) \to \{x\} \times \mathbb{R}^m$) and $proj_2 \circ \psi_\alpha$ is a linear isomorphism from the v.sp. $p^{-1}(x)$, for any $x \in U_\alpha$, to the v.space $\mathbb{R}^m$.

<consequence of the previous>

- on the intersection $U_\alpha \cap U_\beta$,
  $$\psi_\alpha \circ \psi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^m \longrightarrow (U_\alpha \cap U_\beta) \times \mathbb{R}^m$$
  is of the form $(x, v) \longmapsto (x, g_{\alpha\beta}(x)\, v)$
  where $g_{\alpha\beta}(x)$ ("transition function") is a smooth function on $U_\alpha \cap U_\beta$ with values in invertible $m\times m$ matrices.

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for TM, g_{uv} is the Jacobian matrix $\left(\frac{\partial \tilde{x}_i}{\partial x_j}\right)$

for T^*M g_{uv} is its inverse transpose $\left(\frac{\partial x_j}{\partial \tilde{x}_i}\right)$

<u>Ex:</u> $E = M \times \mathbb{R}^m$ - "trivial" vector bundle

- Morphism of vector bundles $\begin{matrix} E \\ p\downarrow \\ M \end{matrix}$, $\begin{matrix} E' \\ \downarrow p' \\ N \end{matrix}$ is a pair of maps $f: M \to N$, $F: E \to E'$ s.t.

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

(i.e. $F: p^{-1}(x) \to p'^{-1}(f(x))$) and $\forall x \in M$ F gives a <u>linear map</u> $F|_{p^{-1}(x)} : p^{-1}(x) \longrightarrow p'^{-1}(f(x))$.

Vector fields as derivations

X - vector field $\to X_x \in T_x^{Alg}$ $\forall x \in M$ is a mapping $X: C^\infty(M) \longrightarrow C^\infty(M)$

$$f \longmapsto (x \mapsto X_x f) =: X(f)$$

locally: $X(f)(x) = \overbrace{\sum_i y_i(x) \left(\frac{\partial}{\partial x_i}\right)_x}^{X} (f) = \sum_i y_i(x) \frac{\partial f}{\partial x_i}(x)$

- smooth; Leibnitz property: $\boxed{X(fg) = f \cdot X(g) + g \cdot X(f)}$. $(*)$

Linear transformations $X: C^\infty(M) \to C^\infty(M)$ satisfying $(*)$ are called <u>derivations</u> of the ring $C^\infty(M)$.

* Derivations of $C^\infty(M)$ = vector fields on M.

<u>Proposition</u>: Let $X: C^\infty(M) \to C^\infty(M)$ be a lin. map which satisfies $(*)$. Then X is a vector field.

<u>Proof</u>: $\forall a \in M$, $X_a(f) = X(f)(a)$ satisfies the conditions of a tangent vector at a. So, X defines a map $X: M \to TM$ with $p \circ X = id_M$. So, locally it can be written as $X_x = \sum_i y_i(x) \left(\frac{\partial}{\partial x_i}\right)_x$. We need to check that $y_i(x)$ are smooth.

$\underset{\substack{\uparrow \\ \text{bump function in} \\ \text{the coord. nbhd}}}{X(x_i \mu)} \overset{=1 \text{ around } a}{=} y_i(x)$ near a. Since X is $C^\infty \to C^\infty$, $y_i(x)$ is C^∞.

$\square$

- <u>Lie bracket of vector fields</u>

Let X, Y two vector fields on M. We can compose them as operators $C^\infty \to C^\infty$.

$$XY(fg) = X(fY(g) + gY(f)) = X(f)Y(g) + fXY(g) + X(g)Y(f) + gXY(f)$$

$$YX(fg) = Y(fX(g) + gX(f)) = Y(f)X(g) + fYX(g) + Y(g)X(f) + gYX(f)$$

$\Rightarrow [X,Y] = XY - YX$ satisfies

$$[X,Y](fg) = f \cdot [X,Y](g) + g[X,Y](f)$$

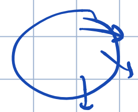
$\Rightarrow [X,Y]$ is a vector field.

def: The Lie bracket of two vector fields X, Y is the vector field $[X,Y]$.

Ex: $M = \mathbb{R}$, $X = f\frac{d}{dx}$, $Y = g\frac{d}{dx}$ $\Rightarrow$ $[X,Y] = (fg' - gf')\frac{d}{dx}$

One-parameter groups of diffeomorphisms

intuition: wind velocity vector field on S^2



moves a particle at x to a new point $\varphi_t(x)$ after time t. after time s, it is at $\varphi_s(\varphi_t(x)) = \varphi_{s+t}(x)$

def: A one-parameter group of diffeomorphisms <"flows"> of a manifold M is a smooth map $\varphi: \mathbb{R} \times M \to M$, $(t, x) \mapsto \varphi_t(x)$ s.t.

- $\varphi_t: M \to M$ is a diffeomorphism
- $\varphi_0 = id$
- $\varphi_{s+t} = \varphi_s \circ \varphi_t$

idea: a vector field gives rise to a 1-param grp of diffeo, under certain assumptions.

- Given a 1-param grp of diffeo φ_t, we can produce a vector field out of it:

for $f \in C^\infty(M)$, set $X_a(f) := \frac{d}{dt}\Big|_{t=0} f(\varphi_t(a))$

- this X_a satisfies Leibnitz property: $X_a(fg) = \frac{\partial}{\partial t}\Big|_{t=0} f(\varphi_t(a))\, g(\varphi_t(a))$
$$= f(a)\frac{d}{dt}\Big|_{t=0} g(\varphi_t(a)) + g(a)\frac{d}{dt}\Big|_{t=0} f(\varphi_t(a))$$
$$= f(a)\, X_a(g) + g(a)\, X_a(f)$$

$\Rightarrow$ X_a is a tangent vector at a.

- Locally: $\varphi_t(x_1, \dots, x_n) = (y_1(t,x), \dots, y_n(t,x))$

$$\frac{d}{dt}\Big|_{t=0} f(y_1, \dots, y_n) = \sum_i \frac{\partial f}{\partial y_i}(y)\underbrace{\frac{dy_i}{dt}(x)\Big|_{t=0}}_{c_i(x)} = \sum_i c_i(x)\frac{\partial f}{\partial x_i}(x) = X(f)$$

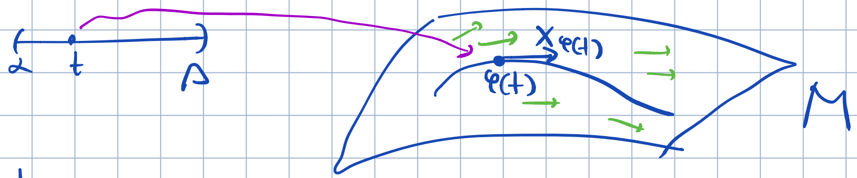
with $X = \sum_i c_i(x)\frac{\partial}{\partial x_i}$ - vector field.

- We want to reverse this process and go from a vector field to the diffeomorphism.
  - First, we want to track the "trajectory of a single particle".

def An <u>integral curve</u> of a vector field X is a smooth map

$\varphi : (\alpha, \beta) \to M$ such that $\boxed{D\varphi_t \left(\frac{d}{dt}\right) = X_{\varphi(t)}}$ $(*)$

$\quad\ \cap$
$\quad\mathbb{R}$



<u>**Example**</u> $M = \mathbb{R}^2$ with coordinates (x,y)

and $X = \frac{\partial}{\partial x}$. The derivative $D\varphi$ of a smooth fun. $\varphi(t) = (x(t), y(t))$

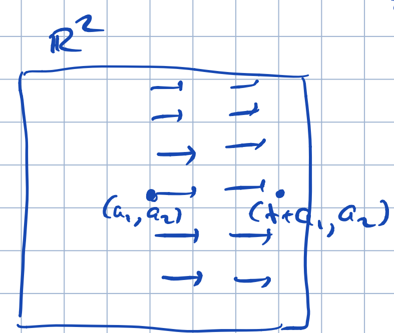
is $D\varphi_t\left(\frac{d}{dt}\right) = \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y}$. So, the equation $(*)$ is:

$$\begin{cases} \frac{dx}{dt} = 1 \\ \frac{dy}{dt} = 0 \end{cases} \Rightarrow \varphi(t) = (t + a_1, a_2)$$

system of ODE

a solution (general)

- particle at (a_1, a_2) is trasported in time t to $(t+a_1, a_2)$



<u>**Theorem**</u>* Given a vector field X on a manifold M and $a \in M$, there exists a <u>maximal</u> integral curve of X with $\varphi(0) = a$.

i.e. the interval (α, β) is maximal

for the proof, we need

10.4 in Hitchin

Thm (Picard-Lindelöf): Let $f(t,x)$ be a continuous function with values in $\mathbb{R}^n$ on $|t - t_0| \le a$, $\|x - x_0\| \le b$ and suppose f satisfies Lipschitz condition $\|f(t,x_1) - f(t,x_2)\| \le L\,\|x_1 - x_2\|$. (for some L "Lipschitz constant")

If $\lambda = \sup |f(t,x)|$ and $h = \min(a, b/\lambda)$, then the differential equation $\frac{dx}{dt} = f(t,x)$, $x(t_0) = x_0$ has a unique solution for $|t - t_0| \le h$.

Proof of THM* Consider (U_γ, ψ_γ) - chart around a. Then if $X = \sum_i c_i(x) \frac{\partial}{\partial x_i}$, the eq. $D\varphi_t\left(\frac{d}{dt}\right) = X_{\varphi(t)}$ can be written as a sys. of ODEs

$\frac{dx_i}{dt} = c_i(x_1, \dots, x_n)$. By Picard-Lindelöf, $\exists!$ sol. on some interval with init. cond. $(x_1(0), \dots, x_n(0)) = \psi_\gamma(a)$.

Suppose $\varphi(\alpha, \beta) \to M$ is any integral curve with $\varphi(0) = a$. $\forall x \in (\alpha, \beta)$, interval $[0, x]$ is compact $\Rightarrow$ can be covered by a fin. number of coord. charts, in each of which we can apply P-L to intervals $[0, \alpha_1], [\alpha_1, \alpha_2], \dots, [\alpha_n, x]$.

Uniqueness $\Rightarrow$ these local solutions agree with φ on any sub-interval containing 0.

$\rightsquigarrow$ then we take ~~the~~ maximal interval on which we can define φ.

$\square$