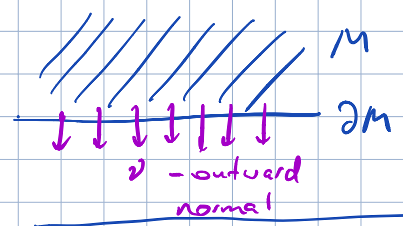
LAST TIME

• If M is an oriented manifold with boundary, then ∂M is oriented:

$$\underbrace{\omega_{\partial M}}_{\text{boundary or. form}} = \iota_\nu \underbrace{\omega_M}_{\text{bulk or. form}}\Big|_{\partial M}$$



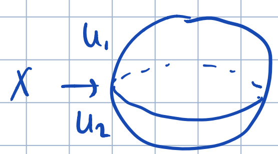
Stokes' Thm: $\displaystyle\int_M d\alpha = \int_{\partial M} \alpha \quad , \quad \alpha \in \Omega_c^{n-1}(M)$

M: oriented, with bdry

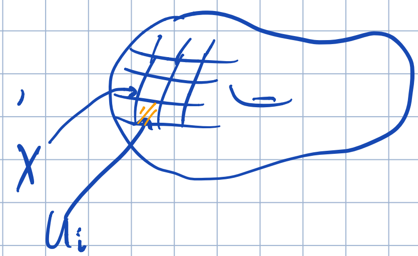
∂M: with induced orientation

Remark: if $M = \coprod_i \underbrace{U_i}_{\text{open}} \cup X$ ← finite union of submanifolds of dim $< n$

then $\displaystyle\int_M \alpha = \sum_i \int_{U_i} \alpha$



,

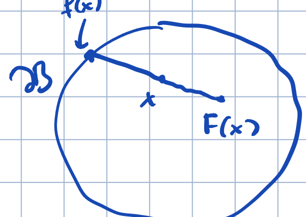


Brouwer fixed point theorem

Let $B = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$, $F: B \to B$ smooth. Then: F has a fixed point $\exists x \in B$ s.t. $F(x) = x$.

Proof: assume F has no fixed pts.

$F(x) \neq x \quad \forall x \in B$.



$$f: B \to \partial B \quad \text{smooth map}$$
$$x \mapsto f(x)$$

$$f(x) = x \quad \text{if} \quad x \in \partial B.$$

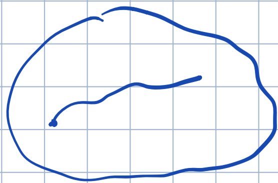
Let ω be the orientation $(n-1)$-form normalized so that $\displaystyle\int_{\partial B} \omega = 1$.

$\partial B = S^{n-1}$

$$1 = \int_{\partial B} \omega = \int_{\partial B} f^*\omega \underset{\text{Stokes'}}{=} \int_B d(f^*\omega) = \int_B f^*(\underbrace{d\omega}_{\text{0 as an }n\text{-form on }S^{n-1}}) = 0 \quad \text{Contradiction!}$$

$\square$

• Given a p-dimensional submanifold $N \overset{\iota}{\hookrightarrow} M$ and a p-form α on M can integrate $\int_{\mathrm{im}(N)} \alpha := \int_N \iota^* \alpha$.



$\gamma : [0,1] \to M$

---

de Rham cohomology in top degree

Theorem If M is cpt, orientable, connected n-mfd, then

$H^n(M) = \mathbb{R}$

$\alpha \in \Omega^n_{cl}(M) \longrightarrow \int_M \alpha$

If $\int_M \omega_M = 1$ $[\omega_M]$ - natural basis in $H^n(M)$

$F : M \to N$ $F^*[\omega_N] = k [\omega_M]$

cpt conn. oriented n-mlds

real numbers

in fact: $k \in \mathbb{Z}$.

Degree of a map

Theorem: Let M, N oriented, cpt, connected n-mlds and $F: M \to N$ a sm. map. Then there exists an integer, the degree of F, s.t.

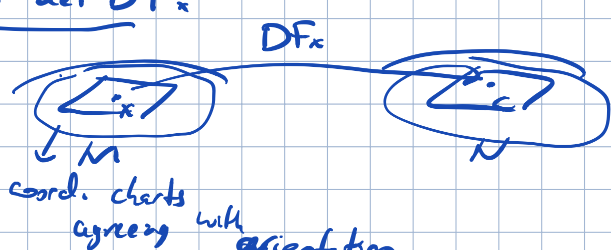
• if $\alpha \in \Omega^n(N)$ then $\int_M F^* \alpha = \deg F \cdot \int_N \alpha$

• if c is a regular value of F, then

$$\deg F = \sum_{x \in F^{-1}(c)} \text{sign} \det DF_x$$

$F^* \omega_N = \lambda \omega_M$

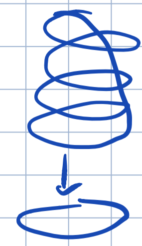
sign $\lambda(x)$



Corollary: if F is not surjective then $\deg F = 0$

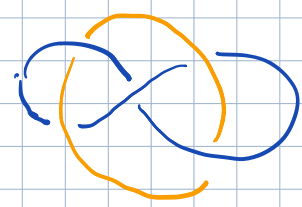
Ex: • if F is an orientation-preserving diffeomorphism $\deg F = 1$

• $F: S^1 \to S^1$, $z \mapsto z^k$ $\qquad \deg F = k$



• $f_1, f_2 : S^1 \to \mathbb{R}^3$ two smooth maps

$\underset{K_1}{\operatorname{im} f_1}$, $\underset{K_2}{\operatorname{im} f_2}$ - assumed to be disjoint



Consider a map $F: S^1 \times S^1 \to S^2$

$$(s,t) \longmapsto \frac{f_1(s) - f_2(t)}{\|f_1(s) - f_2(t)\|}$$

$\deg F$ = "linking number" of K_1 and K_2

Poincaré duality for de Rham cohomology.

__Thm__ Let M be a cpt oriented n-mfld. Then one has a __non-degenerate__ bilinear form

$$H^p(M) \times H^{n-p}(M) \longrightarrow \mathbb{R} \qquad 0 \le p \le n$$

$$(\underset{\Omega^p_{cl}(M)}{[\alpha]}, \underset{\Omega^{n-p}_{cl}(M)}{[\beta]}) \longmapsto \int_M \alpha \wedge \beta$$

This bilinear form gives an iso. $H^p(M) \xrightarrow{\cong} (H^{n-p}(M))^*$

__Corollary__ $\underbrace{\dim H^p(M)}_{p\text{-th Betti number of } M} = \dim H^{n-p}(M)$

relation between $H^1(M)$ and $\pi_1(M)$.

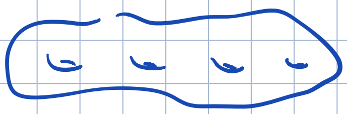
• there is a pairing $\pi_1(M, x_0) \times H^1(M) \longrightarrow \mathbb{R}$

$$([\gamma], [\alpha]) \longmapsto \int_\gamma \alpha := \int_{S^1} \gamma^* \alpha$$

it induces a __nondeg pairing__ $\overbrace{(\underbrace{\pi_1(M)^{ab}}_{\mathbb{Z}^k \oplus \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots} \otimes \mathbb{R})}^{\mathbb{R}^k} \times H^1(M) \longrightarrow \mathbb{R}$

$$G^{ab} = G/\underset{xyx^{-1}y^{-1}}{[G,G]}$$

Ex: $M = \Sigma_g$ $\qquad \pi_1(\Sigma_g)^{ab} \simeq \mathbb{Z}^{2g} \to H^1(\Sigma_g) \simeq \mathbb{R}^{2g}$



$M = X_k$ $\qquad \pi_1(X_k)^{ab} = \mathbb{Z}^{k-1} \oplus \mathbb{Z}_2 \qquad H^1(X_k) = \mathbb{R}^{k-1}$.

k-fold proj $\qquad \pi_1(X_k)^{ab} \otimes \mathbb{R} = \mathbb{R}^{k-1}$