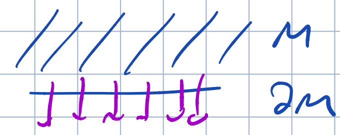
LAST TIME:

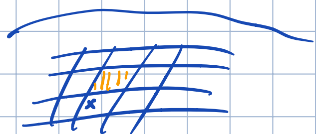
- If M oriented mfd with bdry, then ∂M is oriented: $\omega_{\partial M} = \iota_\nu\, \omega_M \big|_{\partial M}$

bdry or. form   bulk or. form



- Stokes' theorem: $\displaystyle\int_M d\alpha = \int_{\partial M} \alpha$, $\alpha \in \Omega_c^{n-1}(M)$

oriented with bdry   with induced orientation



* the contribution of a tiny parallelogram/parallelipiped to the integral:

$\approx \alpha_x(v_1, \dots, v_n)$ $v_1, \dots, v_n \in T_x M$

* if $M = M_1 \cup M_2$ and $M_1 \cap M_2$ is a finite union of lower-dimensional submanifolds of M

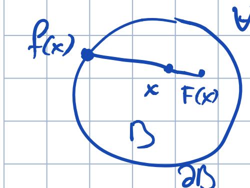
then $\displaystyle\int_M \alpha = \int_{M_1} \alpha|_{M_1} + \int_{M_2} \alpha|_{M_2}$

<corollary of Stokes'>

THM (Brouwer fixed point theorem)

Let $B = \{x \in \mathbb{R}^n \mid \|x\| \le 1\}$ be the unit ball and let $F: B \to B$ be a smooth map. Then F has a fixed point (i.e. $\exists x \in B$ s.t. $F(x) = x$).

Proof: assume F has no fixed point: $F(x) \neq x \ \forall x \in B$.



$\forall x$, extend the line segment $\overline{F(x)x}$ until it meets the bdry ∂B at a pt. $f(x)$.

We have a smooth function $f: B \to \partial B$ s.t. for $x \in \partial B$, $f(x) = x$

Let ω be the standard orientation $(n-1)$-form on $\partial B \simeq S^{n-1}$ with $\displaystyle\int_{\partial B} \omega = 1$

Then $\displaystyle 1 = \int_{\partial B} \omega \underset{\substack{\uparrow \\ \text{since } f|_{\partial B} = id}}{=} \int_{\partial B} f^*\omega \underset{\substack{\uparrow \\ \text{Stokes'}}}{=} \int_B d(f^*\omega) = \int_B f^*(\underbrace{d\omega}_{=0 \,\in\, \Omega^n(S^{n-1})}) = 0$ – contradiction! $\square$

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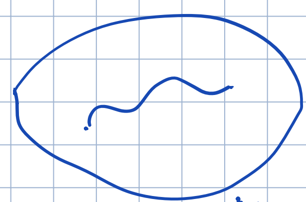
* Given a p-dimensional embedded submanifold $N \overset{\iota}{\hookrightarrow} M$ and a $\underline{p}$-form α on M we can form $\displaystyle\int_N \iota^*\alpha$ – the integral of a p-form over a p-dimensional submanifold

E.g. one can take $\gamma : [0,1] \longrightarrow M$ a smooth path then one can integrate a 1-form along it.



de Rham cohomology in top dimension

~~**Lemma** 8.1**:** let $U^n = \{x \in \mathbb{R}^n \mid |x_i| < 1\}$ (n-open cube) and let $\alpha \in \Omega^n(\mathbb{R}^n)$ with $\operatorname{supp}\alpha \subset U^n$ such that $\int_{U^n} \alpha = 0$. Then $\exists\, \beta \in \Omega^{n-1}(\mathbb{R}^n)$ with $\operatorname{supp}\beta \subset U^n$ s.t. $\alpha = d\beta$.~~

~~(proof: see Hitchin)~~

skip

Theorem If M is compact connected orientable n-manifold, then

$$\boxed{H^n(M) \cong \mathbb{R}}$$

Proof: Cover M by coord. nbhds $\{U_\alpha, \varphi_\alpha : U_\alpha \to \mathbb{R}^n\}_{\alpha \in I}$ with $\varphi_\alpha(U_\alpha) = U^n$ open cube

choose $\{\psi_i\}$ a subordinate partition of unity.

M compact $\Rightarrow$ can assume that we have finitely many charts $U_1, \dots, U_N$.

Using a bump function, fix an n-form α_0 with $\operatorname{supp} \subset U_1$ and with $\int_M \alpha_0 = 1$

(II)

$[\alpha_0] \neq 0 \in H^n(M)$ by Stokes!

Want to show that $\forall \alpha \in \Omega^n(M)$, $[\alpha] = c[\alpha_0]$ or equivalently $\boxed{\alpha = c\alpha_0 + d\gamma.}$

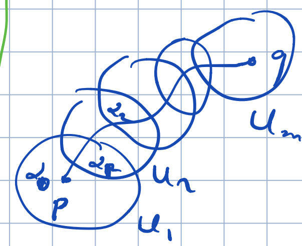
write $\alpha = \sum_i \varphi_i \alpha$ - by linearity, it suffices to prove (#) for $\underbrace{\varphi_i \alpha}$

supported in U_m

skip the proof

M connected $\Rightarrow$ can connect $p \in U_1$ and $q \in U_m$ by a path.

Renumbering U_i's we can assume that the path is covered by a sequence of U_i's: $p \in U_1$, $U_i \cap U_{i+1} \neq \emptyset$, $q \in U_m$



for $1 \leq i \leq m-1$, choose $\alpha_i \in \Omega^n$ with $\operatorname{supp} \subset U_i \cap U_{i+1}$, with $\int \alpha_i = 1$

On U_1: $\int_{U_1} \alpha_0 - \alpha_1 = 0 \underset{\text{Lemma}}{\Rightarrow} \alpha_0 - \alpha_1 = d\beta_1$

continuing: $\alpha_1 - \alpha_2 = d\beta_2$ ... $\alpha_{m-2} - \alpha_{m-1} = d\beta_{m-1}$

$\underset{\text{adding}}{\Rightarrow}$ $\alpha_0 - \alpha_{m-1} = d\left(\sum_{i=1}^{m-1} \beta_i\right)$ (##)

On U_m: $\int \alpha = c = c\int \alpha_{m-1} \underset{\text{Lemma}}{\Rightarrow} \alpha - c\alpha_{m-1} = d\beta$

$\Rightarrow \alpha = c\alpha_{m-1} + d\beta = c\alpha_0 + d(\beta - c\sum_i \beta_i)$ $\boxtimes$

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if $\int_M \omega_n = 1$, $[\omega_n]$ - natural basis in $H^n(M)$ ($\uparrow$ cpt, or, conn)

$F: M \to N$ $\quad F^*: H^n(N) \to H^n(M)$

$F^*[\omega_N] = k[\omega_M]$ $\quad k \in \mathbb{R}$

in fact: $k \in \mathbb{Z}$!

Degree of a map

Thm Let M, N be oriented, compact, connected manifolds of same dimension n and $F: M \to N$ a smooth map. Then there exists an integer, called the degree of F s.t.

- if $\alpha \in \Omega^n(N)$, then $\boxed{\int_M F^*\alpha = \deg F \int_N \alpha}$

- if c is a regular value of F then

$$\boxed{\deg F = \sum_{x \in F^{-1}(c)} \operatorname{sign} \det DF_x}$$

$F^*\omega_N = \lambda\,\omega_M$

$\operatorname{sign} \lambda(x)$

Corollary: if F is not surjective, then $\deg F = 0$

Ex: if F is an orientation preserving diffeo, then $\deg F = 1$.

Ex: $F: \overset{k\text{-sheet}}{S^1} \longrightarrow S^1$ $\qquad \deg F = k$.

$\qquad z \longmapsto z^k$

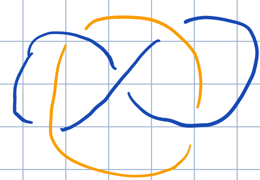
Ex: $f_1, f_2 : S^1 \longrightarrow \mathbb{R}^3$ two smooth maps suppose $\overset{K_1}{\overset{\shortparallel}{\operatorname{im} f_1}}$ is disjoint from $\overset{K_2}{\overset{\shortparallel}{\operatorname{im} f_2}}$

- two circles in $\mathbb{R}^3$ ("knots")

Consider $F: S^1 \times S^1 \longrightarrow S^2$

$$(s,t) \longmapsto \frac{f_1(s) - f_2(t)}{\|f_1(s) - f_2(t)\|}$$

$\deg F$ = "linking number" of K_1 and K_2.



Poincaré duality for de Rham cohomology

Theorem:
Let M be compact, oriented, ~~connected~~ n-manifold. Then

One has a non-degenerate bilinear form $H^p(M) \times H^{n-p}(M) \longrightarrow \mathbb{R}$, $0 \le p \le n$

$$([\underset{\Omega^p_{cl}}{\underset{\cap}{\alpha}}], [\underset{\Omega^{n-p}_{cl}}{\underset{\cap}{\beta}}]) \longmapsto \int_M \alpha \wedge \beta.$$

> this bilinear form gives an iso. $H^p(M) \xrightarrow{\cong} \left(H^{n-p}(M)\right)^*$

Corollary: $\dim H^p(M) = \dim H^{n-p}(M)$.

* Relation between $H^1(M)$ and $\pi_1(M)$:
  - there is a pairing $\pi_1(M, x_0) \times H^1(M) \longrightarrow \mathbb{R}$

$$([\gamma], \alpha) \longmapsto \int_\gamma \alpha = \int_{S^1} \gamma^* \alpha$$

it induces a $\overset{\text{well-defined}}{\text{nondegenerate}}$ pairing $\big(\underbrace{\pi_1(M,x_0)^{ab}}\otimes\mathbb{R}\big)\times H^1(M)\to\mathbb{R}$

abelianization of a group: $G/[G,G]$

commutator subgroup

Thus: $\boxed{H^1(M)\simeq\left(\pi_1^{ab}(M)\otimes\mathbb{R}\right)^*}$

Ex: $\pi_1^{ab}(\Sigma_g)\simeq\mathbb{Z}^{2g}\quad\rightsquigarrow\quad H^1(\Sigma_g)\simeq\mathbb{R}^{2g}$

$\pi_1(X_k)\simeq\mathbb{Z}^{k-1}\oplus\mathbb{Z}_2\quad\rightsquigarrow\quad H^1(X_k)\simeq\mathbb{R}^{k-1}$

k-fold projective space