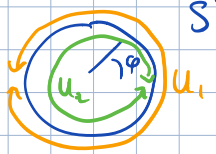
LAST TIME

- $F: M\times[0,1] \to N$ smooth $\Rightarrow F_0^* = F_1^*: \Omega^p(N) \to \Omega^p(M)$
- Poincaré Lemma: $H^p(\mathbb{R}^n) = \begin{cases} \mathbb{R} &, p=0 \\ 0 &, p>0 \end{cases}$
- if $U \subset \mathbb{R}^n$ a "star-shaped region", same answer
- $H^p(M \times \mathbb{R}^n) \simeq H^p(M)$

* **Example:** $H^1(S^1)$



$S^1 = \{ e^{i\varphi} \in \mathbb{C} \}$ parameterize by the angle $\varphi \in \mathbb{R}/_{2\pi}\mathbb{Z}$

$\mu = d\varphi$ is a ~~nowhere-vanishing~~ 1-form on S^1.

μ defined by this f-la in $U_1 = S^1 \setminus \{-1\}$ where $\varphi \in (-\pi, \pi)$ and $U_2 = S^1 \setminus \{+1\}$ where $\varphi \in (0, 2\pi)$

$d\mu = 0$ obviously (e.g. for degree reason)
$\Rightarrow \mu$ closed

assume $\mu = d\underbrace{f}_{\text{global function on } S^1}$ $\quad S^1$ compact $\Rightarrow f$ must have a min and a max
$\Rightarrow df$ must vanish somewhere
but μ is nonvanishing! $\Rightarrow \mu$ is not exact.

So: $H^1(S^1) \neq 0$
and contains the nonzero class $[\mu]$

Let $\alpha \in \Omega^1(S^1)$ any form
$\alpha = \underbrace{g(\varphi)}_{\text{periodic function of } \varphi} d\varphi$

want $\alpha = dh \Rightarrow g(\varphi) = h'(\varphi)$

solution $h(\varphi) = \int_0^{\varphi} ds\, g(s) \quad (+C)$

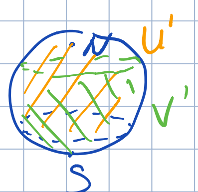
this solution is periodic if $\int_0^{2\pi} ds\, g(s) = 0$

generally: $g(\varphi) = \underbrace{\frac{1}{2\pi}\int_0^{2\pi} ds\, g(s)}_{g_0 \text{ - mean value of } g \text{ on } S^1} + \tilde{g}(\varphi)$ with $\tilde{g}(\varphi) = h'(\varphi)$, $\tilde{g}(\varphi) = g(\varphi) - g_0$, $h(\varphi) = \int_0^{\varphi} ds\, \tilde{g}(\varphi)$

So: $\alpha = g_0 \underbrace{\mu}_{d\varphi} + \underbrace{dh}_{\text{exact form}} \quad \Rightarrow \quad H^1(S^1) = \mathrm{Span}[\mu] = \mathbb{R}.$ ✓

Theorem [6.10]: For $n > 0$, $H^p(S^n) \cong \begin{cases} \mathbb{R} & \text{if } p = 0 \text{ or } n \\ 0 & \text{otherwise} \end{cases}$

Proof Let $n > 1$ (case $n = 1$ was discussed above). Let $1 < p < n$.
Let $\alpha \in \Omega^p_{closed}(S^n)$



$U = S^n \setminus S \xrightarrow{\varphi_U} \mathbb{R}^n$
$V = S^n \setminus N \xrightarrow{\varphi_V} \mathbb{R}^n$
stereographic charts on S^n

$\alpha|_U = du$ $\leftarrow$ since $H^p(\mathbb{R}^n) = 0$)
$\alpha|_V = dv$ for some $u \in \Omega^{p-1}(U)$, $v \in \Omega^{p-1}(V)$

On $U \cap V$: $0 = \alpha|_{U\cap V} - \alpha|_{U \cap V} = du - dv = d(u - v)$

$\Rightarrow \quad u-v \in \Omega^{p-1}_{closed}(\underbrace{U\cap V}_{})$
$\qquad\qquad \underset{diffeo}{\cong} \mathbb{R}\times S^{n-1}$

$$H^{p-1}(\mathbb{R}\times S^{n-1}) = H^{p-1}(S^{n-1}) \underset{\text{by induction}}{=} 0 \quad \text{for} \quad \underline{\underline{1<p<n}}$$

$\Rightarrow u-v = d\omega \;,\; \omega\in\Omega^{p-2}(U\cap V)$

Let $U' = \varphi_U^{-1}(B_2(0))$
$V' = \varphi_V^{-1}(B_2(0))$

Let ψ = bump function with support on $U\cap V$ s.t. $\psi = 1$ on $U'\cap V'$

$\psi\cdot\omega$ - global $(p-2)$-form on S^n (extended by zero)

define $\beta = \begin{cases} u & \text{on } U' \\ v + d(\psi\omega) & \text{on } V' \end{cases}$ - global $(p-1)$-form (restrictions to $U'\cap V'$ agree)

then $d\beta = \alpha \quad \Rightarrow \quad \alpha$ is exact! $\quad \Rightarrow \quad H^p(S^n) = 0$
for $1<p<n$

- If $p=1$, $\quad u-v\in\Omega^0_{closed}(U\cap V)$
$= c$ - a constant function (using that $U\cap V$ is connected, for $n>1$)

$\Rightarrow \beta = \begin{cases} u & \text{on } U \\ v+c & \text{on } V \end{cases}$ - global function, $d\beta = \alpha$
(agree on overlap)

- If $p=n$, $\quad u-v$ defines a class in $H^{p-1}(U\cap V) \cong H^{p-1}(S^{n-1}) \underset{\text{induction}}{\cong} \mathbb{R}$

Let $H^{p-1}(S^{n-1}) = \mathrm{Span}[\omega]$, $\quad \omega \in \Omega^{p-1}_{closed}(S^{n-1})$

$$U\cap V \xrightarrow[\cong]{\Phi} S^{n-1}\times\mathbb{R} \xrightarrow{\pi} S^{n-1}$$

So: $u - v = \lambda\Phi^*\omega + d\omega$, some $\lambda\in\mathbb{R}$. If $\lambda = 0$, then we do as above and find a global $(p-1)$ form β s.t. $\alpha = d\beta$.

- λ is linear in α
and independent of the choice of u, v (shifting them by an exact term can be absorbed into ω)

$\Rightarrow \dim H^n(S^n) \leq 1$

- Need to find α with nonzero λ. Set $\alpha = \Phi^*(\psi(t)\,dt\wedge\omega) \in \Omega^n(S^n)$

ψ - bump function on $\mathbb{R}$

extended by zero outside $\mathrm{supp}\,\Phi^*\psi$

$u = \Phi^*\left(\left(\int_{-\infty}^{t}\psi(s)\,ds\right)\omega\right) \in \Omega^{p-1}(U)$ s.t. $du = \alpha$

extension by zero from $U\cap V$ to U

$v = \Phi^*\left(-\int_t^{\infty}\psi(s)\,ds\right)\omega) \in \Omega^{p-1}(V)$

$$u - v = \underbrace{\left(\int_{-\infty}^{\infty}\psi(s)\,ds\right)}_{\lambda>0}\Phi^*\omega \quad \text{on } U\cap V$$

$\square$

Integration of forms

Orientation Recall a change of variables formula in a multiple integral

$$\int f(y_1,\dots,y_n)\,dy_1\dots dy_n = \int f(y_1(x)\cdots y_n(x))\left|\det \frac{\partial y_i}{\partial x_j}\right| dx_1\dots dx_n \qquad (*)$$

compare to: the change of coords for an n-form on an n-mfd:

$$\theta = f(y_1,\dots,y_n)\,dy_1\wedge\dots\wedge dy_n = f(y_1(x),\dots,y_n(x))\underbrace{\left(\sum_{i_1}\frac{\partial y_1}{\partial x_{i_1}}dx_{i_1}\right)\wedge\dots\wedge\left(\sum_{i_n}\frac{\partial y_n}{\partial x_{i_n}}dx_{i_n}\right)}$$

$$= f(y_1(x),\dots,y_n(x))\cdot\det\left(\frac{\partial y_i}{\partial x_j}\right)\cdot dx_1\wedge\dots\wedge dx_n \qquad \sum_{\sigma\in S_n}\operatorname{sign}\sigma\cdot\frac{\partial y_1}{\partial x_{\sigma(1)}}\cdots\frac{\partial y_n}{\partial x_{\sigma(n)}}\,dx_1\wedge\dots\wedge dx_n$$

So: the difference is the absolute value $|\det|$ in $(*)$. If we can deal with that, we should be able to assign a coord-independent value to $\int_M \underset{\uparrow\, n\text{-form}}{\alpha}$.

... Dealing with $|\det|$ vs $\det$ = orientation ...

def An n-mfd is <u>orientable</u> if it has an <u>everywhere non-vanishing</u> n-form ω.

def Let M be an orientable n-manifold. An <u>orientation</u> on M is an equiv. class of non-vanishing n-forms ω where $\omega\sim\omega'$ if $\omega' = f\omega$ with $f>0$.

- A connected orientable mfd has two orientations $[\pm\omega]$.

Ex 1: $f:\mathbb{R}^{n+1}\to\mathbb{R}$, $\quad \underset{\text{regular value of } f}{c\in}$ $\qquad M = f^{-1}(c)\subset\mathbb{R}^{n+1}$ submanifold (by regular value levelset thm)

if $\left.\frac{\partial f}{\partial x_i}\right|_a \neq 0$, then $x_1,\dots,\hat{x}_i,\dots,x_{n+1}$ are loc. coords on M near a.

On such a patch, consider $\omega = (-1)^i\frac{1}{\frac{\partial f}{\partial x_i}}dx_1\wedge\dots\widehat{dx_i}\dots\wedge dx_{n+1}$ $\quad (\#)$ - non-vanishing n-form

if $\left.\frac{\partial f}{\partial x_j}\right|_a\neq 0$ also, then $\quad \omega = (-1)^j\frac{1}{\partial f/\partial x_j}dx_1\wedge\dots\widehat{dx_j}\dots\wedge dx_{n+1}$ $\overset{\text{substitute}}{\Longleftarrow}$ $dx_j = -\frac{1}{\frac{\partial f}{\partial x_j}}\left(\frac{\partial f}{\partial x_i}dx_i + \dots\right)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \sum_j \frac{\partial f}{\partial x_j}dx_j = 0$ on M

$\Rightarrow$ $(\#)$ defines for all charts a non-vanishing n-form

$\Rightarrow$ M is orientable

E.g. $M = S^n$ with $\omega = (-1)^i\frac{1}{x_i}dx_1\wedge\dots\widehat{dx_i}\dots\wedge dx_{n+1}$ $\quad$ (@)

<u>More generally</u>: if $f:\mathbb{R}^{n+m}\to\mathbb{R}^m$, $M = f^{-1}(c)$, can find $\omega\in\Omega^n(M)$ s.t. $\forall x\in M$

$$\underbrace{df_1\wedge\dots\wedge df_m}_{\in\Lambda^m\left(T_x^*\mathbb{R}^{n+m}/T_x^*M\right)}\wedge\underbrace{\omega}_{\in\Lambda^n T_x^*M} = dx_1\wedge\dots\wedge dx_{n+m}$$

using $\Lambda^m \underset{V}{U}\otimes\Lambda^n V = \Lambda^{n+m}V$

Ex 2 Consider $\mathbb{RP}^n$, $\quad p: S^n \longrightarrow \mathbb{RP}^n$

$$\begin{array}{c} v \\ \text{unit vector} \\ \text{in } \mathbb{R}^{n+1} \end{array} \longmapsto \operatorname{Span}\{v\}$$

~~if $x_1 \neq 0$, use $(x_2 \dots x_{n+1})$ coordinates on S^n, $\left(\frac{x_2}{x_1}, \dots, \frac{x_{n+1}}{x_1}\right)$ coords. on $\mathbb{RP}^n$~~

~~$p(x) = \frac{x}{\sqrt{1-\|x\|^2}}$ smooth with smooth inverse $q(y) = \frac{y}{\sqrt{1+\|y\|^2}}$~~

Let $\sigma : S^n \to S^n$ be the diffeo $\sigma(x) = -x$. Then

$$\underset{(@)}{\sigma^* \omega} = (-1)^i \frac{1}{-x_i} d(-x_1) \wedge \dots \widehat{d(-x_i)} \dots \wedge d(-x_{n+1}) = (-1)^{n+1} \omega$$

Suppose $\mathbb{RP}^n$ is orientable. Then it has a nonvanishing n-form Θ $\Rightarrow$ $p^*\Theta$ is a nonvanishing n-form on S^n and so $p^*\Theta = f\omega$, for f a nonvanishing function.

$$p \circ \sigma = p \Rightarrow f\omega = p^*\Theta = \sigma^* p^* \Theta = \sigma^*(f\omega)$$
$$= (\sigma^* f) \cdot (-1)^{n+1} \omega$$

Thus, if $n = 2m$ even, then $f = -\sigma^* f$, i.e. $f(a) = -f(-a)$

So, if $f(a) > 0$, then $f(-a) < 0$. But S^n is connected, so this means that f must vanish somewhere – contradiction!

$\Rightarrow$ $\mathbb{RP}^{2m}$ is non-orientable!

Proposition A manifold is orientable iff it has a covering by coord. charts such that $\det\left(\frac{\partial y_i}{\partial x_j}\right) > 0$ on the intersection.

Proof Assume M is orientable, ω a non-vanishing n-form. In a coord. chart, $\omega = f(x_1, \dots, x_n)\, dx_1 \wedge \dots \wedge dx_n$. After possibly making a coord. change $x_1 \mapsto c - x_1$, we have coords. s.t. $f > 0$.

Look at two such overlapping charts:

$$\omega = g(y_1, \dots, y_n)\, dy_1 \wedge \dots \wedge dy_n = g(y_1(x), \dots, y_n(x)) \left(\det \frac{\partial y_i}{\partial x_j}\right) dx_1 \wedge \dots \wedge dx_n$$
$$= f(x_1, \dots, x_n)\, dx_1 \wedge \dots \wedge dx_n$$

since $f > 0$, $g > 0$, we have $\det > 0$.

Conversely: suppose we have such coords on $\{U_\alpha\}$. Let $\{\varphi_\alpha\}$ be a partition of unity subordinate to the cover $\{U_\alpha\}$

Set $\omega = \sum_\alpha \varphi_\alpha \, dy_1^\alpha \wedge \dots \wedge dy_n^\alpha$

On a chart U_β with coords $x_1, \dots, x_n$, we have

$$\omega|_{U_\beta} = \sum_\alpha \underbrace{\varphi_\alpha}_{\geq 0} \underbrace{\det\left(\frac{\partial y_i^\alpha}{\partial x_j}\right)}_{>0} dx_1 \wedge \dots \wedge dx_n \qquad \text{- non-vanishing!}$$

$\boxtimes$

· ~~Integration~~ <u>Integration</u> suppose M is orientable and we have chosen an orientation.

We will define the integral $\boxed{\int_M \theta}$ of any n-form θ of compact support on M.

· choose $\{U_\alpha\}$ - coordinate covering with coordinates s.t. $\omega|_{U_\alpha} = h_\alpha \, dx_1 \wedge \dots \wedge dx_n$, $h_\alpha > 0$ ← orientation form

✓ $\theta|_{U_\alpha} = f_\alpha(x_1, \dots, x_n) \, dx_1 \wedge \dots \wedge dx_n$

let $\{\varphi_i\}$ - partition of unity subordinate to $\{U_\alpha\}$

$$\varphi_i \theta|_{U_\alpha} = \underbrace{g_i(x_1, \dots, x_n)}_{\text{smooth function with compact support on the entire } \mathbb{R}^n} dx_1 \wedge \dots \wedge dx_n$$

$\operatorname{supp} \varphi_i \subset U_\alpha$

Define
$$\int_M \theta := \left(\sum_i \int_M \varphi_i \theta := \right) \sum_i \int_{\mathbb{R}^n} g_i(x_1, \dots, x_n) \, dx_1 \cdots dx_n$$

~~(can choose $\{U_\alpha\}$ s.t. they give a finite cover of supp θ)~~

- $\operatorname{supp} \theta$ compact $\Rightarrow \varphi_i \theta \neq 0$ for finitely many i's $\Rightarrow$ finitely many terms are nonzero

+ $\{\operatorname{supp} \varphi_\alpha\}$ are locally finite

(finitely many $\operatorname{supp} \varphi_i$'s intersect $\operatorname{supp} \theta$)

(can cover $\operatorname{supp} \theta$ by nbhds V_{x_j} (fin. many) s.t. each overlaps with fin. many $\operatorname{supp} \varphi_i$'s)

$\int_M \theta$ is well-defined because of the change of variables f-la for integration + consistent choice of sign of det Jac from orientation.

Independence on partition of unity:

[Guillemin - Pollack, p. 167-168]

- if $\operatorname{supp} \theta \subset U_\alpha$ - single coord chart

then $\int_M \theta = \sum_i \int_M \varphi_i \theta$, because $\sum \varphi_i = 1$

$\quad \| $

$\int_{\mathbb{R}^n} f_\alpha(x_1 \dots x_n) \, dx_1 \dots dx_n$

~~compact~~

~~$\{U_\alpha\}$ - finite cover of supp θ (by coord. charts on M)~~

- if $\{\varphi_i\}$, $\{\varphi_j'\}$ two partitions of unity,

$$\int_M \underbrace{\varphi_i \theta}_{\text{supported in some } U_\alpha} = \sum_j \int_M \varphi_j' \varphi_i \theta \quad \Rightarrow \left[\int_M \theta\right]_{\{\varphi_i\}} = \sum_i \int_M \theta \varphi_i = \sum_{i,j} \int_M \theta \varphi_i \varphi_j' = \sum_j \int_M \theta \varphi_j' = \left[\int_M \theta\right]_{\{\varphi_j'\}}$$

"compactly supported"

Properties: ① $\int_M : \Omega_c^n(M) \longrightarrow \mathbb{R}$ is a linear map, i.e.

$$\int(\alpha+\beta) = \int\alpha + \int\beta \quad , \int c\alpha = c\int\alpha$$

② changing orientation on M results in changing the sign of $\int_M \theta$

③ if $F : M \longrightarrow N$ is an orientation-preserving diffeomorphism and $\theta \in \Omega_c^n(N)$

then $$\int_N \theta = \int_M F^*\theta$$