LAST TIME:

- Orientation on $^{n\text{-manifold}}M$ = a nowhere-vanishing n-form ω on M up to equivalence $\omega \sim f\omega$, $f > 0$

Ex: S^n is orientable
$\mathbb{RP}^{2m}$ is non-orientable

Proposition A manifold is orientable iff it has a covering by coord. charts such that $\det\left(\frac{\partial y_i}{\partial x_j}\right) > 0$ on the intersection.

Proof Assume M is orientable, ω a non-vanishing n-form. In a coord. chart, $\omega = f(x_1, \dots, x_n)\, dx_1 \wedge \dots \wedge dx_n$. After possibly making a coord. change $x_1 \mapsto c - x_1$, we have coords. s.t. $f > 0$.

Look at two such overlapping charts:

$$\omega = g(y_1, \dots, y_n)\, dy_1 \wedge \dots \wedge dy_n = g(y_1(x), \dots, y_n(x)) \left(\det \frac{\partial y_i}{\partial x_j}\right) dx_1 \wedge \dots \wedge dx_n$$
$$= f(x_1, \dots, x_n)\, dx_1 \wedge \dots \wedge dx_n$$

since $f > 0$, $g > 0$, we have $\det > 0$.

Conversely: suppose we have such coords. Let $\{\varphi_\alpha\}$ be a partition of unity subordinate to $^{\text{the cover}}\{U_\alpha\}$ on $\{U_\alpha\}$

Set $\omega = \sum_\alpha \varphi_\alpha \, dy_1^\alpha \wedge \dots \wedge dy_n^\alpha$

On a chart U_β with coords $x_1, \dots, x_n$, we have

$$\omega|_{U_\beta} = \sum_\alpha \underbrace{\varphi_\alpha}_{\geq 0} \underbrace{\det\left(\frac{\partial y_i^\alpha}{\partial x_j}\right)}_{>0} dx_1 \wedge \dots \wedge dx_n \quad \text{- non-vanishing!}$$

$\blacksquare$

· ~~Integration~~ **Integration** suppose M is orientable and we have chosen an orientation.

We will define the integral $\boxed{\int_M \theta}$ of any n-form θ of compact support on M.

· choose $\{U_\alpha\}$ - coordinate covering ✓ with coordinates s.t. $\omega|_{U_\alpha} = h_\alpha \, dx_1 \wedge \dots \wedge dx_n$, $h_\alpha > 0$ ← orientation form

$\theta|_{U_\alpha} = f_\alpha(x_1, \dots, x_n) \, dx_1 \wedge \dots \wedge dx_n$

let $\{\varphi_i\}$ - partition of unity subordinate to $\{U_\alpha\}$

$$\varphi_i \theta|_{U_\alpha} = \underbrace{g_i(x_1, \dots, x_n)}_{\text{smooth function with compact support on the entire } \mathbb{R}^n} dx_1 \wedge \dots \wedge dx_n$$

$\operatorname{supp} \varphi_i \subset U_\alpha$

Define $$\int_M \theta := \left(\sum_i \int_M \varphi_i \theta := \right) \sum_i \int_{\mathbb{R}^n} g_i(x_1, \dots, x_n) \, dx_1 \cdots dx_n$$

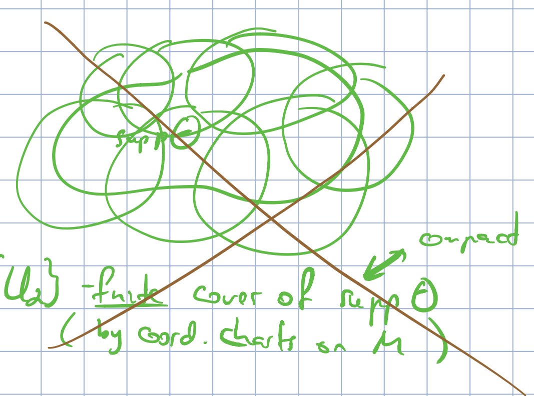
~~(can choose $\{U_\alpha\}$ s.t. they give a finite cover of supp θ)~~

• supp θ compact + $\{\operatorname{supp} \varphi_\alpha\}$ are locally finite $\Rightarrow \varphi_i \theta \neq 0$ for finitely many i's $\Rightarrow$ finitely many terms are nonzero

(finitely many supp φ_i's intersect supp θ)

(can cover supp θ by nbhds V_{x_j} (fin. many) s.t. each overlaps with fin. many supp φ_i's)

$\int_M \theta$ is well-defined because of the change of variables f-la for integration + consistent choice of sign of det Jac from orientation.



Independence on partition of unity:

[Guillemin - Pollack, p. 167-168]

• if supp $\theta \subset U_\alpha$ - single coord chart

then $\int_M \theta = \sum_i \int_M \varphi_i \theta$, because $\sum \varphi_i = 1$

$$\int_M \theta = \int_{\mathbb{R}^n} f_\alpha(x_1, \dots, x_n) \, dx_1 \cdots dx_n$$

• if $\{\varphi_i\}$, $\{\varphi_j'\}$ two partitions of unity,

$$\int_M \underbrace{\varphi_i \theta}_{\text{supported in one } U_\alpha} = \sum_j \int_M \varphi_j' \varphi_i \theta \quad \Rightarrow \left[\int_M \theta\right]_{\{\varphi_i\}} = \sum_i \int_M \theta \varphi_i = \sum_{i,j} \int_M \theta \varphi_i \varphi_j' = \sum_j \int_M \theta \varphi_j' = \left[\int_M \theta\right]_{\{\varphi_j'\}}$$

"compactly supported"

Properties: ① $\int_M : \Omega_c^n(M) \longrightarrow \mathbb{R}$ is a linear map, i.e.

$$\int(\alpha+\beta) = \int\alpha + \int\beta, \quad \int c\alpha = c\int\alpha$$

② changing orientation on M results in changing the sign of $\int_M \theta$

③ if $F : M \longrightarrow N$ is an orientation-preserving diffeomorphism and $\theta \in \Omega_c^n(N)$

then $$\int_N \theta = \int_M F^*\theta$$

Stokes' Theorem

simple version:

compact support

Theorem: Let M be an oriented n-manifold and $\alpha \in \Omega_c^{n-1}(M)$. Then

$$\boxed{\int_M d\alpha = 0}$$

Proof Choose a partition of unity $\{\varphi_i\}$ subordinate to a coord. cover $\{U_\alpha\}$

$\alpha = \sum_i \varphi_i \alpha$. In a coord. nbhd,

$$\varphi_i \alpha = a_1\, dx_2 \wedge \dots \wedge dx_n - a_2\, dx_1 \wedge dx_3 \wedge \dots \wedge dx_n + \dots$$

and $$d(\varphi_i \alpha) = \left(\frac{\partial a_1}{\partial x_1} + \dots + \frac{\partial a_n}{\partial x_n}\right) dx_1 \wedge \dots \wedge dx_n$$

$$\Rightarrow \int_M \varphi_i \alpha = \int_{\underset{\text{coord nbhd containing supp } \varphi_i}{U_\alpha}} \varphi_i \alpha \overset{\text{def. of } \int}{=} \int_{\mathbb{R}^n} \left(\frac{\partial a_1}{\partial x_1} + \dots + \frac{\partial a_n}{\partial x_n}\right) dx_1 \cdots dx_n \qquad (*)$$

Consider $$\int_{\mathbb{R}^n} \frac{\partial a_1}{\partial x_1} dx_1 \cdots dx_n \overset{\text{Fubini}}{=} \int_{\mathbb{R}} dx_n \int_{\mathbb{R}} dx_{n-1} \cdots \underbrace{\int_{\mathbb{R}} dx_1 \frac{\partial a_1}{\partial x_1}}_{\lim_{N\to\infty} a_1\big|_{-N}^{N} = 0} = 0$$

fund thm. of calculus

since a_1 has compact support, thus $a_1\big|_{x_1 = \pm N} = 0$ for N large enough.

other terms in $(*)$ vanish similarly. ∎

Proposition Let M be a compact orientable n-manifold.
Then $H^n(M) \neq 0$.

Proof. M orientable $\to$ let ω be a nonvanishing n-form. ω is closed ($d\omega = 0$ for degree reason).
Consider $[\omega] \in H^n(M)$.
Choose the orientation defined by ω:

$$\int_M \omega = \sum_i \int \underbrace{f_i}_{\geq 0} \, dx_1 \cdots dx_n > 0$$

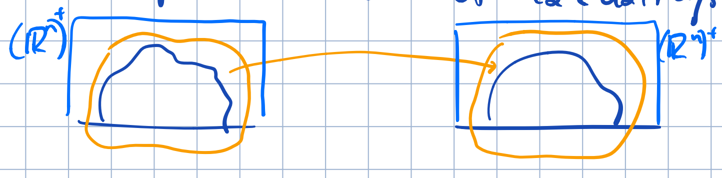
each f_i is positive somewhere

If $\omega = d\alpha$, then $\int_M \omega = \int_M d\alpha = 0$ by Thm $\Rightarrow$ ω is not exact
$\Rightarrow [\omega] \neq 0 \in H^n(M)$ $\boxtimes$

- **Manifolds with boundary**

def An n-dimensional manifold with boundary is a top. space M with a collection of open sets $\{U_\alpha\}$ ($\subset M$) and maps $\varphi_\alpha : U_\alpha \to (\mathbb{R}^n)^+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$ s.t.

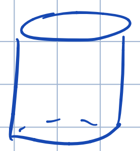
- $M = \bigcup_\alpha U_\alpha$
- φ_α is a homeomorphism between U_α ($\subset M$) and an open set $\varphi_\alpha(U_\alpha) \subset (\mathbb{R}^n)^+$
- $\forall \alpha, \beta$, $\varphi_\beta \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is the restriction of a smooth map from a nbhd of $\varphi_\alpha(U_\alpha \cap U_\beta) \subseteq (\mathbb{R}^n)^+ \subset \mathbb{R}^n$ to $\mathbb{R}^n$



Boundary ∂M of M is defined as $\partial M = \{x \in M \mid \varphi_\alpha(x) \in \{(x_1, \ldots, x_{n-1}, 0)\} \in \mathbb{R}^n\}$
These charts define the structure of an $(n-1)$-mfd on ∂M.

Ex:
① $(\mathbb{R}^n)^+$ (model space) is a manifold with boundary $x_n = 0$
② Closed unit ball $\{x \in \mathbb{R}^{n+1} \mid \|x\| \leq 1\}$ is a mfd w/ bdry S^{n-1}

3. cylinder $I \times S^1$ is a mfd with bdry $S^1 \sqcup S^1$



Diff. forms on a mfd w/ bdry: locally, they are restrictions of smooth forms on some open set in $\mathbb{R}^n$ to $(\mathbb{R}^n)^+$.

A form on M restricts to a form on the boundary.

Proposition If a mfd. M with bdry is oriented, there is an induced orientation on ∂M.

Proof Choose loc. coord. systems on M s.t. ∂M is given by $x_n = 0$ and $\det \frac{\partial y_i}{\partial x_j} > 0$.

On overlapping nbhds, $y_i = y_i(x_1, \dots, x_n)$, $y_n(x_1, \dots, x_{n-1}, 0) = 0$

The Jacobian matrix has the form at $x_n = 0$

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & & \vdots \\ 0 \quad\cdots\quad 0 & & \frac{\partial y_n}{\partial x_n} \end{pmatrix} \qquad (\#)$$

$\varphi_\beta \varphi_\alpha^{-1}$ maps $\underset{x_n>0}{(\mathbb{R}^n)^+}$ to $\underset{y_n>0}{(\mathbb{R}^n)^+}$

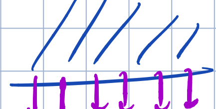
on $\underset{x_n=0}{\mathbb{R}^{n-1}\times\{0\}}$ to $\underset{y_n=0}{\mathbb{R}^{n-1}\times\{0\}}$ $\Rightarrow$ $\frac{\partial y_n}{\partial x_n} > 0$

$$\det \mathrm{Jac}_{\partial M} \cdot \underbrace{\left.\frac{\partial y_n}{\partial x_n}\right|_{x_n=0}}_{>0} = \underbrace{\det \mathrm{Jac}_n}_{>0} \Rightarrow \det \mathrm{Jac}_{\partial M} > 0$$

$\Rightarrow$ ∂M is orientable, with orientation $(n-1)$-form $\sum_\alpha \varphi_\alpha \, dx_1^\alpha \wedge \dots dx_{n-1}^\alpha \, (-1)^n$ $\quad\square$

φ_α: part. of unity on ∂M

Rem: locally, induced orientation form on the bdry:



$\nu = -\frac{\partial}{\partial x_n}$ "outward normal"

$$\underset{\text{bulk orientation}}{dx_1 \wedge \dots \wedge dx_n} \longrightarrow \iota_\nu (dx_1 \wedge \dots \wedge dx_n)$$
$$= (-1)^n dx_1 \wedge \dots \wedge dx_{n-1}$$

– boundary orientation

Stokes' Theorem   Let M be an n-dimensional oriented manifold with boundary ∂M. Let $\alpha \in \Omega^{n-1}_c(M)$ - form with compact support.

Then, using the induced orientation on ∂M:

$$\boxed{\int_M d\alpha = \int_{\partial M} \alpha}$$