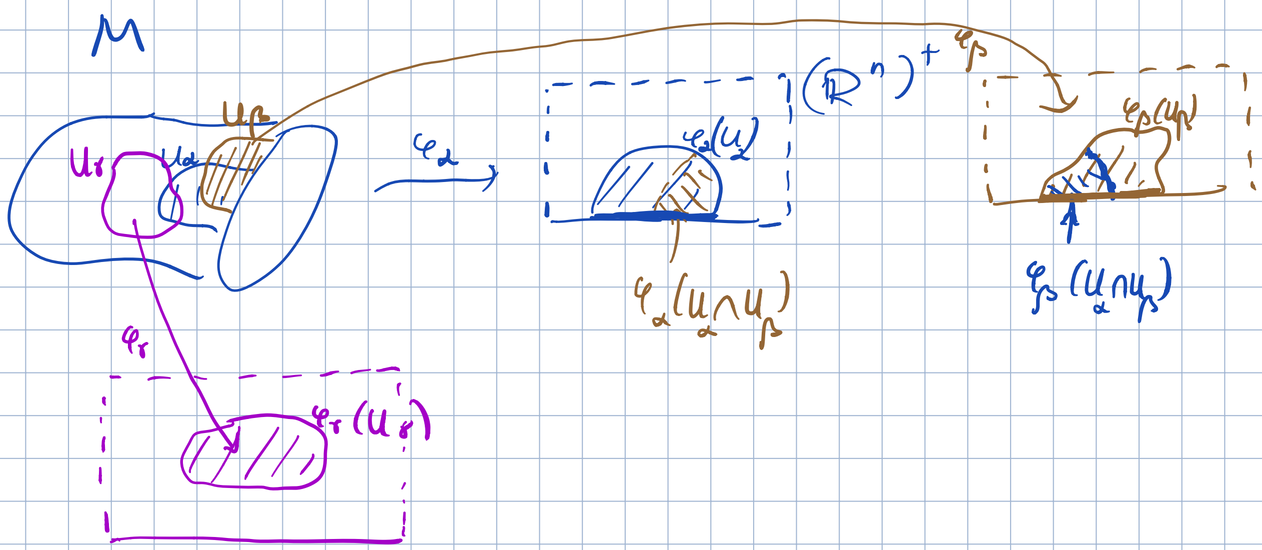


Organizational

> Exam: Mon 11/16, 10^{00} am – Wed 11/18, 10^{00} am

> CIFs: window open until 11:59 pm, 11/15

LAST TIME:

- $\theta \in \Omega^n_c(M)$ (M oriented) $\rightsquigarrow \displaystyle\int_M \theta := \sum_i \underbrace{\int_M \varphi_i \theta}_{\text{calculated as ordinary multiple integral in a coord. chart}} = \sum_i \int_{\mathbb{R}^n} f_i(x_1 \ldots x_n)\,dx_1 \ldots dx_n$

  (φ_i: partition of unity)

x $v_1 \ldots v_n \in T_x$ $\quad \Sigma\, \theta(v_1, \ldots, v_n)$

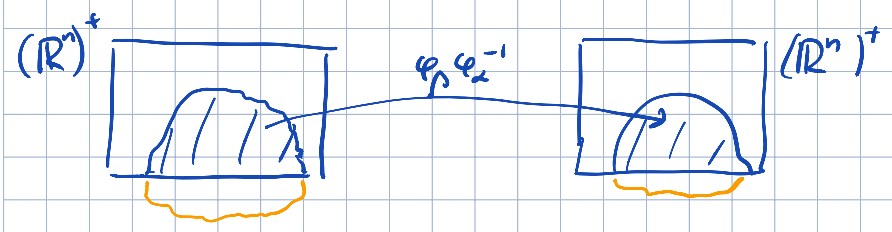
- $\theta \in \Omega^{n-1}_c(M) \Rightarrow \displaystyle\int_M d\theta = 0$

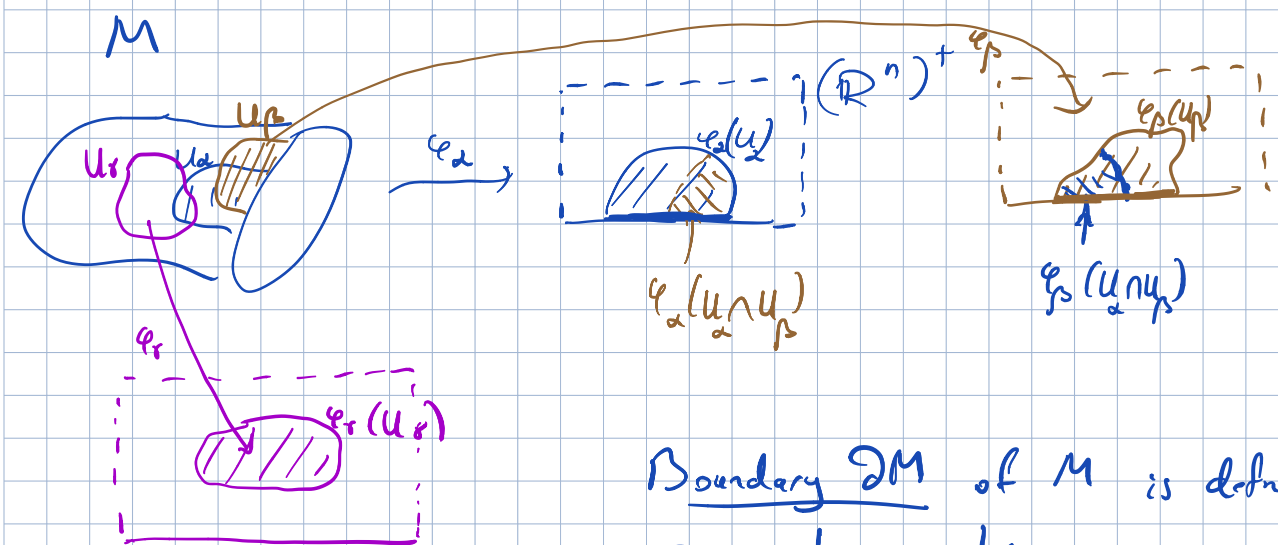
- Corollary: if M cpt, oriented, then $H^n(M) \neq 0$.

Manifolds with boundary

def: An n-mfd with bdry is a top. space M with a collection of open sets $\{U_\alpha\}$ ($\subset M$) and maps $\varphi_\alpha : U_\alpha \to (\mathbb{R}^n)^+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$ s.t.

- $M = \bigcup_\alpha U_\alpha$
- φ_α is a homeomorphism between U_α ($\subset M$) and an open set $\varphi_\alpha(U_\alpha) \subset (\mathbb{R}^n)^+$
- $\forall \alpha, \beta$, $\varphi_\beta \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is a restriction of a smooth map from a nbhd of $\varphi_\alpha(U_\alpha \cap U_\beta)$ in $\mathbb{R}^n$ to a nbhd of $\varphi_\beta(U_\alpha \cap U_\beta)$ in $\mathbb{R}^n$.





Boundary ∂M of M is defined as

$$\partial M = \{x \in M \mid \varphi_\alpha(x) = (x_1, \dots, x_{n-1}, 0)\}$$

($\in \mathbb{R}^n$)

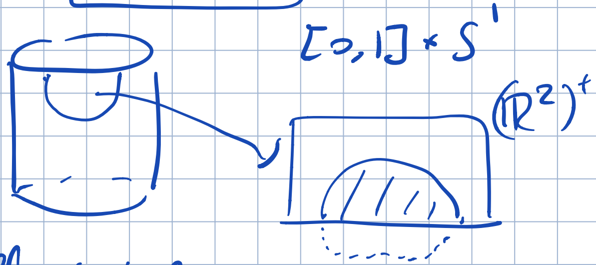
These charts define the structure of an $(n-1)$-mfd on ∂M.

Ex: ① $(\mathbb{R}^n)^+$ is a mfd with bdry

② closed ball $M = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ is a mfd with bdry

$\partial M = S^{n-1}$

③ $M = \boxed{I \times S^1}$ $\qquad \partial M = S^1 \amalg S^1$



Differential forms on a mfd w/ bdry: locally, they are restrictions of smooth forms on some open set in $\mathbb{R}^n$ to $(\mathbb{R}^n)^+$

<u>Proposition</u> If M is an oriented mfd with bdry, then there is an induced orientation on the bdry ∂M.

<u>Proof</u>: Choose a loc. coord. system on M s.t. ∂M is given by $x_n = 0$ and $\det \frac{\partial y_i}{\partial x_j} > 0$. On an overlap $y_i = y_i(x_1 \dots x_n)$, $y_n(x_1, \dots, x_{n-1}, 0) = 0$
$i = 1 \dots n-1$

$$\mathrm{Jac}\big|_{x_n=0} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \\ \vdots & & \vdots \\ 0 \ \cdots \ 0 & & \frac{\partial y_n}{\partial x_n} \end{pmatrix}$$

$$0 < \det \mathrm{Jac} = \underset{\substack{\vee \\ 0}}{\frac{\partial y_n}{\partial x_n}} \det \boxed{\mathrm{Jac}_{\partial M}} \Rightarrow \boxed{\det \mathrm{Jac}_{\partial M} > 0}$$

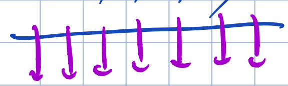
$\Rightarrow \partial M$ is orientable

orientation: $\boxed{\sum_\alpha \varphi_\alpha \, dx_1^\alpha \wedge \dots \wedge dx_{n-1}^\alpha \cdot (-1)^n}$

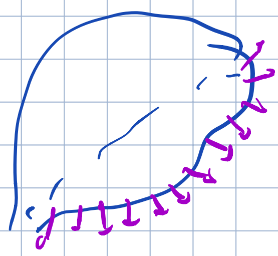
$\uparrow$ part. of unity on ∂M

$\square$

<u>Rem</u>: locally, induced orientation form on ∂M



$\nu = -\frac{\partial}{\partial x_n}$ outward pointing normal vector field (to the bdry)



$dx_1 \wedge \dots \wedge dx_n \longrightarrow \iota_\nu (dx_1 \wedge \dots \wedge dx_n)$

$= (-1)^n dx_1 \wedge \dots \wedge dx_{n-1}$

– boundary orientation form

Stokes theorem

M - n-dimensional mfd with bdry ∂M. Let $\alpha \in \Omega^{n-1}_c(M)$

Then
$$\boxed{\int_M d\alpha = \int_{\partial M} \alpha}$$

with induced orientation

Proof: $\alpha = \sum \varphi_i \alpha \qquad \int_M d\alpha = \sum_i \int_M d(\varphi_i \alpha)$

locally $\varphi_i \alpha = \sum_{i=1}^{n} (-1)^n a_i \, dx_1 \wedge \dots \widehat{dx_i} \dots dx_n$

if $\operatorname{supp} \varphi_i \subset U_\beta \not\ni \partial M \Rightarrow \int_M d(\varphi_i \alpha) = 0$ – know!

& $U_\beta \cap \partial M$

$$\int_M \underline{\underline{d(\varphi_i \alpha)}} = \int_{x_n \geq 0} \left(\frac{\partial a_1}{\partial x_1} + \dots + \underbrace{\frac{\partial a_n}{\partial x_n}} \right) dx_1 \dots dx_n = \int_{\mathbb{R}^{n-1}} \left. a_n \right|_0^\infty dx_1 \dots dx_{n-1}$$

$$= -\int_{\mathbb{R}^{n-1}} a_n(x_1, \dots, x_{n-1}, 0)\, dx_1 \dots x_{n-1} = \int_{\partial M} \varphi_i \alpha$$

$\varphi_i \alpha \big|_{\partial M} = (-1)^{n-1} a_n \, dx_1 \wedge \dots \wedge dx_{n-1}$

& we use the induced or.

$(-1)^n dx_1 \dots dx_n$

$\square$