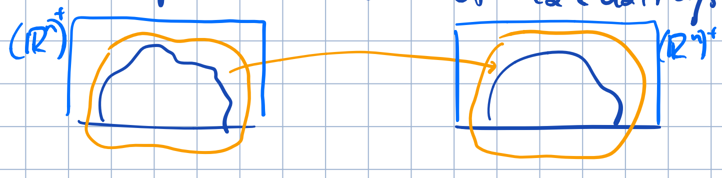
• Manifolds with boundary

def An n-dimensional manifold with boundary is a top. space M with a collection of open sets $\{U_\alpha\} \subset M$ and maps $\varphi_\alpha : U_\alpha \to (\mathbb{R}^n)^+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$ s.t.

- $M = \bigcup_\alpha U_\alpha$
- φ_α is a homeomorphism between $U_\alpha \subset M$ and an open set $\varphi_\alpha(U_\alpha) \subset (\mathbb{R}^n)^+$
- $\forall \alpha, \beta$, $\varphi_\beta \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is the restriction of a smooth map from a nbhd of $\varphi_\alpha(U_\alpha \cap U_\beta) \subseteq (\mathbb{R}^n)^+ \subset \mathbb{R}^n$ to $\mathbb{R}^n$

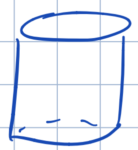


Boundary ∂M of M is defined as $\partial M = \{x \in M \mid \varphi_\alpha(x) \in \{(x_1, \dots, x_{n-1}, 0)\} \in \mathbb{R}^n\}$
These charts define the structure of an $(n-1)$-mfd on ∂M.

Ex: ① $(\mathbb{R}^n)^+$ (model space) is a manifold with boundary $x_n = 0$

② Closed unit ball $\{x \in \mathbb{R}^{n+1} \mid \|x\| \leq 1\}$ is a mfd w/ bdry S^{n-1}

3 cylinder $I \times S^1$ is a mfd with bdry $S^1 \amalg S^1$



Diff. forms on a mfd w/ bdry : locally, they are restrictions of smooth forms on some open set in $\mathbb{R}^n$ to $(\mathbb{R}^n)^+$.

A form on M restricts to a form on the boundary.

Proposition If a mfd. M with bdry is oriented, there is an induced orientation on ∂M.

Proof Choose loc. coord. systems on M s.t. ∂M is given by $x_n = 0$ and $\det \frac{\partial y_i}{\partial x_j} > 0$.

On overlapping nbhds, $y_i = y_i(x_1, \dots x_n)$, $y_n(x_1, \dots, x_{n-1}, 0) = 0$

The Jacobian matrix has the form at $x_n = 0$

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & & \frac{\partial y_1}{\partial x_n} \\ \vdots & & & \vdots \\ & & & \\ 0 & \cdots & 0 & \frac{\partial y_n}{\partial x_n} \end{pmatrix} \qquad (\#)$$

$\varphi_\beta \varphi_\alpha^{-1}$ maps $\underset{x_n > 0}{(\mathbb{R}^n)^+}$ to $\underset{y_n > 0}{(\mathbb{R}^n)^+}$ $\Rightarrow \frac{\partial y_n}{\partial x_n} > 0$

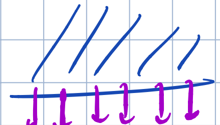
on $\underset{x_n = 0}{\mathbb{R}^{n-1} \times \{0\}}$ to $\underset{y_n = 0}{\mathbb{R}^{n-1} \times \{0\}}$

$$\det \mathrm{Jac}_{\partial M} \cdot \underbrace{\frac{\partial y_n}{\partial x_n}\Big|_{x_n = 0}}_{> 0} = \underset{> 0}{\det \mathrm{Jac}_n} \Rightarrow \det \mathrm{Jac}_{\partial M} > 0$$

$\Rightarrow \partial M$ is orientable, with orientation $(n-1)$-form $\sum_\alpha \varphi_\alpha \, dx_1^\alpha \wedge \dots dx_{n-1}^\alpha \, (-1)^n$ $\boxed{}$

φ_α: part. of unity on ∂M

Rem: locally, induced orientation form on the bdry:



$\nu = -\frac{\partial}{\partial x_n}$ "outward normal"

$\underset{\text{bulk orientation}}{dx_1 \wedge \dots \wedge dx_n} \longrightarrow \iota_\nu (dx_1 \wedge \dots \wedge dx_n)$

$= (-1)^n dx_1 \wedge \dots \wedge dx_{n-1}$

- boundary orientation

Stokes' Theorem

Let M be an n-dimensional oriented manifold with boundary ∂M. Let $\alpha \in \Omega_c^{n-1}(M)$ - form with compact support.

Then, using the induced orientation on ∂M:

$$\boxed{\int_M d\alpha = \int_{\partial M} \alpha}$$

Proof: write $\alpha = \sum_i \varphi_i \alpha$, $\int_M d\alpha = \sum_i \int_M d(\varphi_i \alpha)$

locally: $\varphi_i \alpha = \sum_{i=1}^{n} (-1)^{i-1} a_i \, dx_1 \wedge \dots \widehat{dx_i} \dots \wedge dx_n$ $\quad (*)$

if $\operatorname{supp} \varphi_i \subset U_\beta$ - open set not intersecting the bdry, then $\int_M d(\varphi_i \alpha) = 0$ by the previous version of Stokes' Thm.

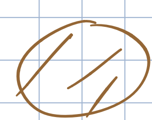
if U_β intersects with ∂M - then:

$$\int_M d(\varphi_i \alpha) = \int_{x_n \geq 0} \left(\frac{\partial a_1}{\partial x_1} + \dots + \frac{\partial a_n}{\partial x_n} \right) dx_1 \dots dx_n = \int_{\mathbb{R}^{n-1}} a_n \Big|_0^\infty dx_1 \dots dx_{n-1}$$

$$= -\int_{\mathbb{R}^{n-1}} a_n(x_1, \dots, x_{n-1}, 0) \, dx_1 \dots dx_{n-1} = \int_{\partial M} \varphi_i \alpha$$

since $\varphi_i \alpha = (-1)^{n-1} a_n \, dx_1 \wedge \dots \wedge dx_{n-1}$ - the last term of $(*)$

and we use the induced orientation $(-1)^n dx_1 \wedge \dots \wedge dx_{n-1}$ ▨

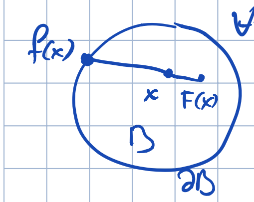


<corollary>

THM (Brouwer fixed point theorem)

Let $B = \{ x \in \mathbb{R}^n \mid \|x\| \leq 1 \}$ be the unit ball and let $F: B \to B$ be a smooth map. Then F has a fixed point (i.e. $\exists x \in B$ s.t. $F(x) = x$).

Proof: assume F has no fixed point: $F(x) \neq x \;\; \forall x \in B$.



$\forall x$, extend the line segment $\overline{F(x)x}$ until it meets the bdry ∂B at a pt. $f(x)$.

We have a smooth function $f: B \to \partial B$ s.t. for $x \in \partial B$, $f(x) = x$

Let ω be the standard orientation $(n-1)$-form on $\partial B \simeq S^{n-1}$ with $\int_{\partial B} \omega = 1$

Then $1 = \int_{\partial B} \omega \underset{\text{since } f|_{\partial B} = id}{=} \int_{\partial B} f^*\omega \underset{\text{Stokes'}}{=} \int_B d(f^*\omega) = \int_B f^*(\underbrace{d\omega}_{=0 \,\in \Omega^n(S^{n-1})}) = 0$ – contradiction! ☐

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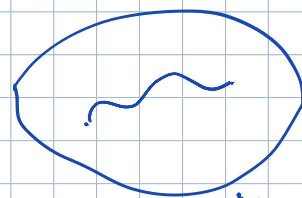
* Given a p-dimensional embedded submanifold $N \overset{\iota}{\hookrightarrow} M$ and a $\underline{p}$-form α on M we can form $\int_N \iota^*\alpha$ – the integral of a p-form over a p-dimensional submanifold

E.g. one can take $\gamma : [0,1] \longrightarrow M$ a smooth path then one can integrate a 1-form along it.



M

de Rham cohomology in top dimension

Lemma 8.1: let $U^n = \{x \in \mathbb{R}^n \mid |x_i| < 1\}$ ← open n-cube and let $\alpha \in \Omega^n(\mathbb{R}^n)$ with $\operatorname{supp} \alpha \subset U^n$ such that $\int_{U^n} \alpha = 0$. Then $\exists\, \beta \in \Omega^{n-1}(\mathbb{R}^n)$ with $\operatorname{supp} \beta \subset U^n$ s.t. $\alpha = d\beta$.

(proof: see Hitchin)

Theorem If M is compact connected orientable n-manifold, then

$$\boxed{H^n(M) \cong \mathbb{R}}$$

Proof: Cover M by coord. nbhds $\{U_\alpha, \varphi_\alpha : U_\alpha \to \mathbb{R}^n\}_{\alpha \in I}$ with $\varphi_\alpha(U_\alpha) = U^n$ open cube choose $\{\varphi_i\}$ a subordinate partition of unity.

M compact $\Rightarrow$ can assume that we have finitely many charts $U_1, \dots, U_N$.

Using a bump function, fix an n-form α_0 with $\operatorname{supp} \subset U_1$ and with $\int_M \alpha_0 = 1$

(II)

$[\alpha_0] \neq 0 \in H^n(M)$ by Stokes!

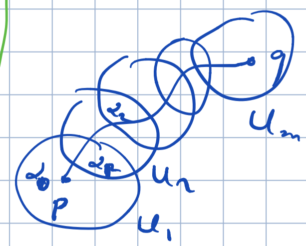
Want to show that $\forall\, \alpha \in \Omega^n(M)$, $[\alpha] = c[\alpha_0]$ or equivalently $\boxed{\alpha = c\alpha_0 + d\gamma.}$ (#)

write $\alpha = \sum_i \varphi_i \alpha$ - by linearity, it suffices to prove (#) for $\underbrace{\varphi_i \alpha}$

supported in U_m

M connected $\Rightarrow$ can connect $p \in U_1$ and $q \in U_m$ by a path.

Renumbering U_i's we can assume that the path is covered by a sequence of U_i's: $p \in U_1$, $U_i \cap U_{i+1} \neq \emptyset$, $q \in U_m$



for $1 \leq i \leq m-1$, choose $\alpha_i \in \Omega^n$ with supp $\subset U_i \cap U_{i+1}$, with $\int \alpha_i = 1$

On U_1: $\int_{U_1} \alpha_0 - \alpha_1 = 0 \underset{\text{Lemma}}{\Longrightarrow} \alpha_0 - \alpha_1 = d\beta_1$

continuing: $\alpha_1 - \alpha_2 = d\beta_2$ ... $\alpha_{m-2} - \alpha_{m-1} = d\beta_{m-1}$

$\underset{\text{adding}}{\Longrightarrow}$ $\alpha_0 - \alpha_{m-1} = d\left(\sum_{i=1}^{m-1} \beta_i\right)$ (##)

On U_m: $\int \alpha = c = c\int \alpha_{m-1} \underset{\text{Lemma}}{\Longrightarrow} \alpha - c\,\alpha_{m-1} = d\beta$

$\Rightarrow \alpha = c\,\alpha_{m-1} + d\beta = c\,\alpha_0 + d\left(\beta - c\sum_i \beta_i\right)$ ☒

Degree of a map

Thm Let M, N be oriented, compact, connected manifolds of same dimension n and $F: M \to N$ a smooth map. Then there exists an integer, called the degree of F s.t.

- if $\alpha \in \Omega^n(N)$, then $\boxed{\int_M F^*\alpha = \deg F \int_N \alpha}$

- if c is a regular value of F then
$$\boxed{\deg F = \sum_{x \in F^{-1}(c)} \operatorname{sign} \det DF_x}$$

Corollary: if F is not surjective, then $\deg F = 0$

Ex: if F is an orientation preserving diffeo, then $\deg F = 1$.

Ex: $F: S^1 \xrightarrow{k\text{-sheet}} S^1$, $z \longmapsto z^k$ $\qquad \deg F = k$.

Ex: $f_1, f_2 : S^1 \longrightarrow \mathbb{R}^3$ two smooth maps suppose $\underset{K_1}{\overset{\shortparallel}{\operatorname{im} f_1}}$ is disjoint from $\underset{K_2}{\overset{\shortparallel}{\operatorname{im} f_2}}$

- two circles in $\mathbb{R}^3$ ("knots")

Consider $F: S^1 \times S^1 \longrightarrow S^2$

$$(s,t) \longmapsto \frac{f_1(s) - f_2(t)}{\| f_1(s) - f_2(t) \|}$$

$\deg F =$ "linking number" of K_1 and K_2.