MATH60330
8/12/20

Lemma** (continuity criterion for maps to a subspace)
Let X, Y top. spaces, $A \subset Y$ (subset) with subspace topology. Then

(a) The inclusion map $i: A \to Y$ is continuous

(b) $f: X \to A$ is cont. iff the composition $X \xrightarrow{f} A \xrightarrow{i} Y$ is cont.

Ex (cont. maps involving subspaces)

1. $$GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$$
$$A \mapsto A^{-1}$$

by Lm**
(suffices to prove continuity of $GL_n \xrightarrow{A^{-1}} GL_n \xhookrightarrow{i} Mat_{n \times n}$ - by Lm* suffices to check continuity component-wise)

2. Let G be one of $SL_n(\mathbb{R})$, $O(n)$, $SO(n)$ with subspace topology as subsets of $M_{n\times n}(\mathbb{R})$.
Then the map $G \to G$, $A \mapsto A^{-1}$ is continuous.
- follows from 1. and:

Lemma If
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ A & & B \end{array}$$
and $f(A) \subset B$, then $f|_A : A \to B$ is continuous w.r.t. subspace topology on A, B.

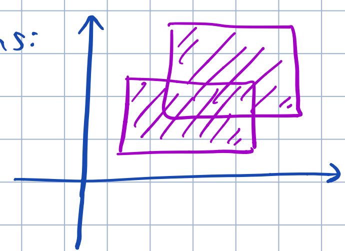
Proof:
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ A & \xrightarrow[f|_A]{} & B \end{array}$$

i, j cont. (Lm^{**} (a)) $\Rightarrow$ $f \circ i$ - cont
$\quad = j \circ f|_A$ - cont $\Rightarrow$
$\xRightarrow{Lm^{**}(b)}$ $f|_A$ - cont.

Product topology. def For X, Y top. spaces, the product topology on the Cartesian product
$X \times Y = \{(x,y) \mid x \in X, y \in Y\}$ is the topology generated by subsets
$\mathcal{B} = \{U \times V \mid U \underset{\text{open}}{\subset} X,\ V \underset{\text{open}}{\subset} Y\}$

Rem. $\mathcal{B}$ is indeed a basis: (a) $\forall (x,y) \in X \times Y \in \mathcal{B}$
(b) $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$

· $\mathcal{B}$ is not itself a topology - not closed under unions:
Ex: $X = Y = \mathbb{R}$
$U_{1,2}, V_{1,2}$ - open intervals



- union is not in $\mathcal{B}$!

Lemma The product topology on $\mathbb{R}^m \times \mathbb{R}^n$ agrees with standard (metric) topology on $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$
<Proof: homework>

Lemma@ (continuity criterion for maps to a product).
Let X, Y_1, Y_2 top. spaces
(i) Projection maps $p_i : Y_1 \times Y_2 \to Y_i$ are continuous
(ii) a map $f : X \to Y_1 \times Y_2$ is continuous iff the compositions
$X \xrightarrow{f} Y_1 \times Y_2 \xrightarrow{p_i} Y_i$ are cont. for $i = 1, 2$

$p_i \circ f$ = "i-th component map" of f.

- This is a generalization of Lm* from last time (which was for maps to $\mathbb{R}^n$)

Lemma# Let $f : X \to Y$ be a map of top. spaces.
Let $\mathcal{B}$ be a basis for topology on Y. Then f is cont.
iff $f^{-1}(B) \underset{\text{open}}{\subset} X \quad \forall B \in \mathcal{B}$.

<Obvious>

Proof of Lm@: (i) Let $U \underset{\text{open}}{\subset} Y_1$. $p_1^{-1}(U) = U \times Y_2 \underset{\text{open}}{\subset} Y_1 \times Y_2 \Rightarrow p_1$ cont.
p_2 - similar

(ii) $\Rightarrow$: $X \xrightarrow{f} Y_1 \times Y_2$ cont. $\Rightarrow p_i \circ f$ cont as a composition (using (i))
$\Leftarrow$: b Lm# it suffices to check that $f^{-1}(U_1 \times U_2) \underset{\text{o}}{\subset} X \quad \forall\, U_1 \underset{\text{open}}{\subset} Y_1,\ U_2 \underset{\text{open}}{\subset} Y_2$

$$f^{-1}(U_1 \times U_2) = \underbrace{f_1^{-1}(U_1)}_{\text{open}} \cap \underbrace{f_2^{-1}(U_2)}_{\text{open}} \underset{\text{open}}{\subset} X \quad \boxed{\checkmark}$$

Lemma Let G be one of groups $GL_n(\mathbb{R}), SL_n(\mathbb{R}), O(n), SO(n)$ with subspace topology as a subset of $M_{n\times n}(\mathbb{R})$.
Then G is a "topological group", i.e. G is a top space and a group, s.t.
(a) multiplication map $G \times G \xrightarrow[\mu]{} G$ is cont.
(b) inversion map $G \longrightarrow G$, $g \longmapsto g^{-1}$ is cont.

Proof (b) - already discussed

(a)
$$\begin{array}{ccc} G\times G & \xrightarrow{\mu} & G \\ \downarrow{\scriptstyle i\times i} & & \downarrow{\scriptstyle i} \\ M_{n\times n}(\mathbb{R})\times M_{n\times n}(\mathbb{R}) & \xrightarrow[m]{} & M_{n\times n}(\mathbb{R}) \end{array}$$

$(*)$ $m \circ \underbrace{(i \times i)}_{\substack{\text{cont. since} \\ \text{its components are}}} = i \circ \mu$

$G\times G \xrightarrow[p_i]{} G \xrightarrow[i]{} M_{n\times n}(\mathbb{R})$

$\Rightarrow (*)$ is cont $\underset{\substack{\text{continuity} \\ \text{of maps to subspace}}}{\Rightarrow}$ μ is cont. $\boxed{\checkmark}$

Quotient topology

def Let X be a top. space and $\sim$ an equiv. relation on X.
Denote $X/\sim$ the set of equiv. classes, $p: X \longrightarrow X/\sim$, $x \longmapsto [x]$ the quotient map.

Quotient topology on $X/\sim$ is the collection of subsets
$$\mathcal{T} = \{ U \subset X/\sim \mid p^{-1}(U) \underset{\text{open}}{\subset} X \}.$$
Set $X/\sim$ with topology $\mathcal{T}$ is the "quotient space."

If $p: \underset{\substack{\uparrow \\ \text{top space}}}{X} \longrightarrow Y$ surjective map, then $Y = X/\sim$ where $x \sim x'$ iff $p(x) = p(x')$
In particular, Y can be equipped with quotient topology.

Examples

1. Let $A \subset X$. Define an equiv. rel. $\sim$ on X: $x \sim y$ if $x = y$ or $x, y \in A$.

   (A: subset; X: top space)

$X/A = X/\sim$.   Ex: D^n/S^{n-1} - it is homeomorphic to S^n (will see later).

1' attaching space: $A \xrightarrow{f\text{ - cont.}} Y$, $A \subset X$ subset $\rightsquigarrow$ $X \cup_f Y = X \amalg Y / a \sim f(a) \ \forall a \in A$ f - "attaching map"

Stopped here

2. real projective space

$\mathbb{RP}^n = \{1\text{-dim. subspaces of } \mathbb{R}^{n+1}\}$

Map $S^n \to \mathbb{RP}^n$, $v \mapsto$ subspace generated by v is surjective $\Rightarrow$ $\mathbb{RP}^n = S^n/(v \sim \pm v)$ with quotient topology

3. complex projective space

$\mathbb{CP}^n = \{1\text{-dim subspaces of } \mathbb{C}^{n+1}\} = S^{2n+1}/(v \sim zv)$, $z \in S^1$

4. Grassman manifold

$G_k(\mathbb{R}^{n+k}) = \{k\text{-dim subspaces of } \mathbb{R}^{n+k}\}$.

We have a surj. map

$$V_k(\mathbb{R}^{n+k}) = \{(v_1, \dots, v_k) \mid v_i \in \mathbb{R}^{n+k}, v_i\text{'s are o/n}\} \twoheadrightarrow G_k(\mathbb{R}^{n+k})$$
$$(v_1, \dots, v_k) \longmapsto \operatorname{Span}\{v_1, \dots, v_k\}$$

Subspace top. on $V_k(\mathbb{R}^{n+k}) \subset \mathbb{R}^{k(n+k)}$ induces a quotient top on $G_k(\mathbb{R}^{n+k})$

Complex Grassmanian $G_k(\mathbb{C}^{n+k})$ - similar construction.

Lemma (continuity criterion for a map out of a quotient space)

(i) Projection map $p: X \to X/\sim$ is continuous

(ii) A map $f: X/\sim \to Y$ is cont. iff the composition $X \xrightarrow{p} X/\sim \xrightarrow{f} Y$ is cont.

<proof: homework>