Examples (of quotient spaces)

LAST TIME: 1. X/A, ...

1': $X \cup_f Y$ - attaching space

$f: A \to Y$ attaching map, $A \subset X$

2. real projective space

$$\mathbb{RP}^n = \{1\text{-dim. subspaces of } \mathbb{R}^{n+1}\}$$

Map $S^n \to \mathbb{RP}^n$, $v \mapsto$ subspace generated by v, is surjective $\Rightarrow$ $\mathbb{RP}^n = S^n/(v \sim \pm v)$ with quotient topology

3. complex projective space

$$\mathbb{CP}^n = \{1\text{-dim subspaces of } \mathbb{C}^{n+1}\} = S^{2n+1}/(v \sim zv), \quad z \in S^1$$

4. Grassman manifold

$$G_k(\mathbb{R}^{n+k}) = \{k\text{-dim subspaces of } \mathbb{R}^{n+k}\}$$

We have a surj. map

$$V_k(\mathbb{R}^{n+k}) = \{(v_1, \dots, v_k) \mid v_i \in \mathbb{R}^{n+k}, v_i\text{'s are o/n}\} \twoheadrightarrow G_k(\mathbb{R}^{n+k})$$

$$(v_1, \dots, v_k) \longmapsto \mathrm{Span}\{v_1, \dots, v_k\}$$

Subspace top. on $V_k(\mathbb{R}^{n+k}) \subset \mathbb{R}^{k(n+k)}$ induces a quotient top on $G_k(\mathbb{R}^{n+k})$

Complex Grassmanian $G_k(\mathbb{C}^{n+k})$ - similar construction.

Lemma (continuity criterion for a map out of a quotient space)

(i) Projection map $p: X \to X/\sim$ is continuous

(ii) A map $f: X/\sim \; \to Y$ is cont. iff the composition

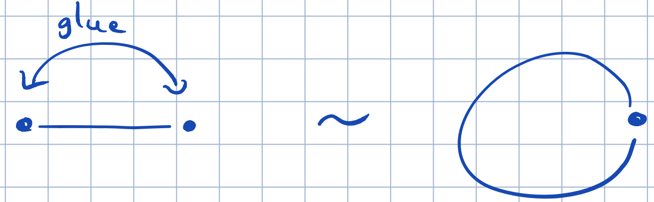
$$X \xrightarrow{p} X/\sim \; \xrightarrow{f} Y \text{ is cont.}$$

<proof: homework>

- under certain assumption, $f: X \xrightarrow{1-1} Y$ a cont. bijection automatically has a ~~contin~~ cont. inverse (i.e. f - homeomorphism) will discuss later

Ex: (1) $[-1,1]/\{\pm 1\} \overset{\text{homeo}}{\sim} S^1$

$f: \quad [t] \longmapsto e^{\pi i t}$ - cont. bijection



- f cont. because $[-1,1] \xrightarrow{p} [-1,1]/\{\pm 1\} \xrightarrow{f} S^1 \xrightarrow{i} \mathbb{C} = \mathbb{R}^2$

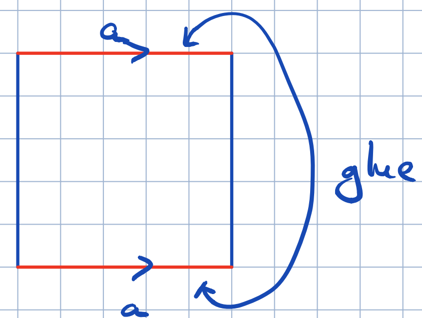
$t \longmapsto e^{\pi i t} = (\cos \pi t, \sin \pi t)$

- cont. fun., since its components are cont.

$i \circ f \circ p$ cont. $\underset{\substack{\text{criterion}\\ \text{of continuity}\\ \text{of map to a subspace}}}{\Rightarrow}$ $f \circ p$ is cont $\underset{\substack{\text{continuity}\\ \text{criterion}\\ \text{for map}\\ \text{out of a quotien}}}{\Longrightarrow}$ f is cont

(2) more generally: $D^n / S^{n-1} \overset{\text{homeo}}{\sim} S^n$

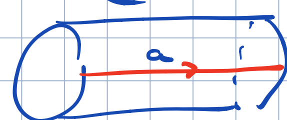
(3) $[-1,1] \times [-1,1] \Big/ \begin{array}{l}(s,-1) \sim (s,1)\\ \forall s \in [-1,1]\end{array}$



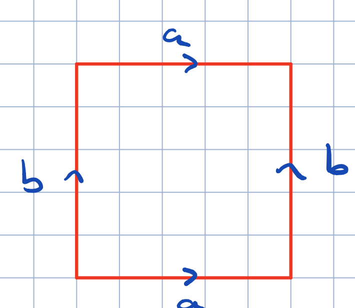
This quotient $\overset{\text{homeo}}{\sim}$ $\overset{\text{cylinder}}{C} = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [-1,1], y^2 + z^2 = 1\}$

$f: [(s,t)] \longmapsto (s, \cos \pi t, \sin \pi t)$

C:

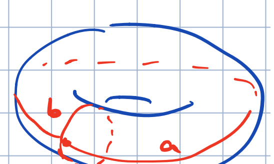


(4) $[-1,1] \times [-1,1] \Big/ \begin{array}{l}(s,-1) \sim (s,1)\\ (-1,t) \sim (1,t)\end{array}$



this quotient $\sim$ $T = \{x \in \mathbb{R}^3 \mid d(x,K) = r\}$, $0 < d < 1$

where $K = \{(x_1, x_2, 0) \mid x_1^2 + x_2^2 = 1\}$



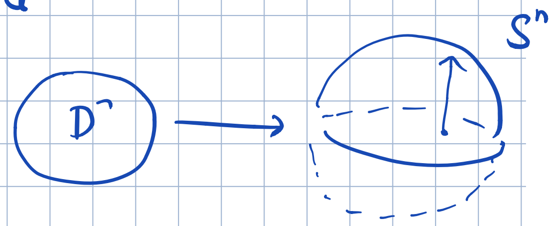
map: $f: [(s,t)] \longmapsto \big(\cos \pi s \cdot (1 + r \cos \pi t), \sin \pi s \cdot (1 + r \cos \pi t), r \sin \pi t\big)$

(5) **Claim** $D^n/_{v\sim -v}$ for $v \in S^{n-1}$ $\overset{\text{homeo}}{\sim}$ $\mathbb{R}P^n = S^n \subset \mathbb{R}^{n+1}/_{u\sim -u}$

define $f: D^n \longrightarrow S^n$

$(x_1,\dots,x_n) \longmapsto (x_1,\dots,x_n,\sqrt{1-x_1^2-\dots-x_n^2})$

- embedding of D^n as the upper hemisphere



Lemma: The induced map $\underline{f}: D^n/_\sim \longrightarrow S^n/_\sim$, $[x] \longrightarrow [f(x)]$ is a (A) well-defined (B) cont. (C) bijection

Proof: (A) $v \in S^{n-1} \overset{f}{\longmapsto} (v,0)$, $-v \in S^{n-1} \overset{f}{\longmapsto} (-v,0)$ — equivalent in S^n $\Rightarrow$ represent the same point in $S^n/_\sim$

$\Rightarrow \underline{f}$ well-defined

(B) f has cont. components $\Rightarrow$ f is continuous

for $\underline{f}$ we have

$$\begin{tikzcd} D^n \arrow[r, "f"] \arrow[d, "p_1"] & S^n \arrow[d, "p_2"] \\ D^n/_\sim \arrow[r, "\underline{f}"] & S^n/_\sim \end{tikzcd}$$

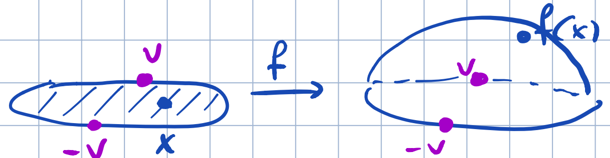
$\underbrace{\overset{\text{cont}}{p_2} \circ \overset{\text{cont}}{f}}_{\text{cont}} = \overset{?}{\underline{f}} \circ \overset{\text{cont}}{p_1}$

$\underset{\text{criterion for cont. for a map out of a quotient}}{\Rightarrow}$ $\underline{f}$ is cont

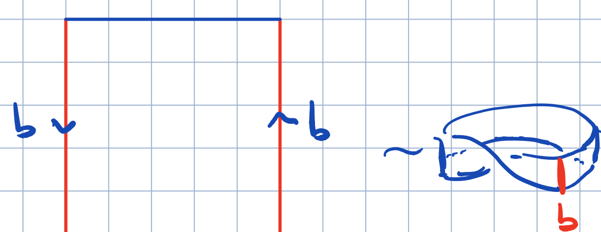
(C) Let HS_+ - upper hemisphere (closed)

HS_+ contains a representative of each equiv. class $\Rightarrow$ $f: D^n \underset{1-1}{\longrightarrow} HS_+$ induces a surjective map $\underline{f}$

- the only equiv. pts in HS_+ are $(u,-u)$ on equator $\Rightarrow$ $\underline{f}$ is injective

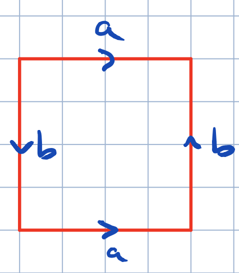


(6) $[-1,1]\times[-1,1]/_{(-1,t)\sim(1,-t)}$

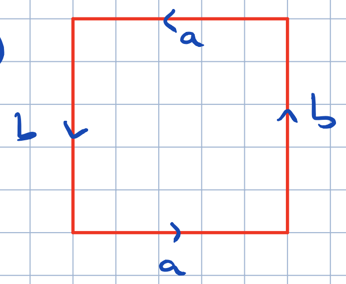


= Möbius band

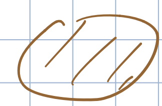
(7) edge identifications



= Klein bottle

(8) 

$\sim \mathbb{R}P^2$



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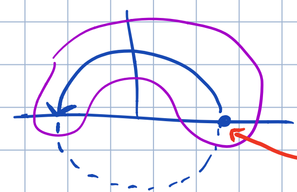
Properties of topological spaces

- if $f: X \to Y$ is a cont. bijection, when is f^{-1} cont. ?

Not always! Ex: $f: [0,1) \longrightarrow S^1 \subset \mathbb{C} = \mathbb{R}^2$

$t \longmapsto e^{2\pi i t}$ - cont. bijection

$g = f^{-1} : S^1 \longrightarrow [0,1)$ is not cont!

$g^{-1}[0,\frac{1}{2}) = f[0,\frac{1}{2}) =$  - not an open subset of S^1!

(because of nbhd of)

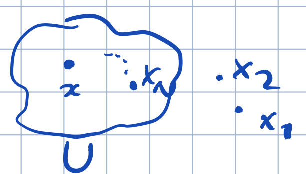
Proposition (criterion for continuity of inverse)

Let $f: X \to Y$ be a cont. bijection. Then f is a homeomorphism provided that X is compact and Y is Hausdorff.

Hausdorff spaces

def Let X top. space, $x_1, x_2, x_3, \ldots \in X$ a sequence in X and $x \in X$.

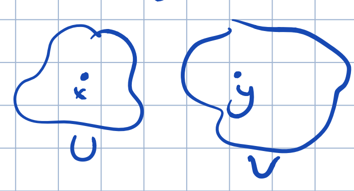
Then x is a limit of x_i's if $\forall\, U \subset X$ ($U \ni x$, open) $\exists N$ s.t. $x_i \in U$ for all $i \geq N$



Ex: • if X a top. space with indiscrete topology, every point is the limit of every sequence!

• There is at most one limit of $\{x_i\}$ if the top space has the following property:

def A top space X is Hausdorff if $\forall\, x,y \in X$, $x \neq y$, there are disjoint open sets $U, V \subset X$, $U \ni x$, $V \ni y$



Lemma: $\mathbb{R}^n$ is Hausdorff. Also, any subspace $U \subset \mathbb{R}^n$ is Hausdorff.

Proof: $x, y \in U$ then $B_r(x), B_r(y)$ are disjoint open nbhds in $\mathbb{R}^n$ of x, y if $r \leq \text{dist}(x, y)$.

$B_r(x) \cap U$, $B_r(y) \cap U$ - disjoint nbhds in U. $\square$.

Lemma: Let X be a top. space and A a closed subspace of X. If $x_n \in A$ is a sequence with limit x, then $x \in A$.

Proof: if $x \notin A$, then $x \in \underbrace{X \setminus A}_{\text{open}}$ $\Rightarrow$ all but finitely many of x_n's belong to $X \setminus A$ - contradiction! $\square$.