Properties of topological spaces

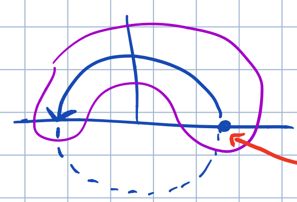
- if $f: X \to Y$ is a cont. bijection, when is f^{-1} cont. ?

Not always! Ex: $f: [0,1) \longrightarrow S^1 \subset \mathbb{C} = \mathbb{R}^2$

$t \longmapsto e^{2\pi i t}$ - cont. bijection

$g = f^{-1}: S^1 \longrightarrow [0,1)$ is not cont!

$g^{-1}[0,\frac{1}{2}) = f[0,\frac{1}{2})$ =



- not an open subset of S^1!

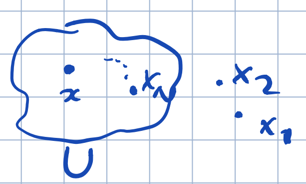
(because of nbhd of)

Proposition* (criterion for continuity of inverse)

Let $f: X \to Y$ be a cont. bijection. Then f is a homeomorphism provided that X is compact and Y is Hausdorff.

Hausdorff spaces

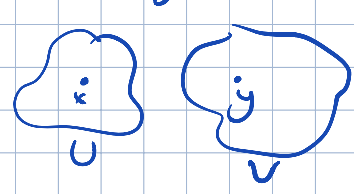
def Let X top. space, $x_1, x_2, x_3, \dots \in X$ a sequence in X and $x \in X$.
Then x is a limit of x_i's if $\forall\, U \subset X$ ($U \ni x$, open) $\exists N$ s.t. $x_i \in U$ for all $i \geq N$



Ex: • if X a top. space with indiscrete topology, every point is the limit of every sequence!

• There is at most one limit of $\{x_i\}$ if the top space has the following property:

def A top space X is Hausdorff if $\forall\, x, y \in X$, $x \neq y$, there are disjoint open sets $U, V \subset X$ ($U \ni x$, $V \ni y$)



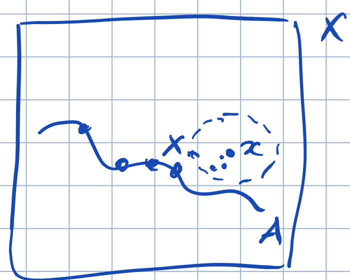
Lemma[@]: $\mathbb{R}^n$ is Hausdorff. Also, any subspace $U \subset \mathbb{R}^n$ is Hausdorff.

Proof: $x, y \in U$ then $B_r(x)$, $B_r(y)$ are disjoint open nbhds in $\mathbb{R}^n$ of x, y if $r \leq \text{dist}(x,y)$.

$B_r(x) \cap U$, $B_r(y) \cap U$ - disjoint nbhds in U. $\boxed{}$.

Lemma: Let X be a top. space and A a closed subspace of X. If $x_n \in A$ is a sequence with limit x, then $x \in A$.

Proof: if $x \notin A$, then $x \in \underbrace{X \setminus A}_{\text{open}}$ $\Rightarrow$ all but finitely many of x_n's belong to $X \setminus A$ - contradiction! $\boxed{}$.

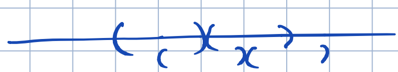


Compact spaces

def An open cover for a top space X is a collection of open sets $U_\alpha \subset X$ s.t. $\bigcup_\alpha U_\alpha = X$

If for every open cover of X there is a finite subcollection that also covers X, then X is called compact.

- Non-example: $\mathbb{R}$ (with stand. topology) $= \bigcup_{n \in \mathbb{Z}} \overbrace{(n-1, n+1)}^{U_n}$



- removing a single U_n the rest covers $\mathbb{R} \setminus \{n\}$
$\Rightarrow$ no finite subcover $\Rightarrow$ $\mathbb{R}$ non-compact!

Lemma[#]: if $f: X \to Y$ cont. map and X compact, then the image $f(X)$ is compact.

- In particular, if X cpt, then any quotient $X/\sim$ is cpt (since one has $p: X \twoheadrightarrow X/\sim$)

Proof: Let $\{U_\alpha\}_{\alpha \in A}$ an open cover of subspace $f(X)$. Then $U_\alpha = \underbrace{V_\alpha}_{\text{some open in } Y} \cap f(X)$

$\{f^{-1}(V_\alpha)\}_{\alpha \in A}$ - open cover of X

X cpt $\Rightarrow$ choose a finite subcover $\{f^{-1}(V_\alpha)\}_{\alpha \in A'}$ of X $\Rightarrow$ $\{U_\alpha\}_{\alpha \in A'}$ - finite cover of $f(X)$

$A' \subset A$ finite subset

$\Rightarrow f(X)$ compact $\boxed{}$

Lemma##: 1. If $\underset{\text{closed subspace}}{K} \subset \underset{\text{compact}}{X}$ then K is compact

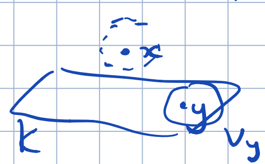
2. If $\underset{\substack{\text{compact}\\\text{subspace}}}{K} \subset \underset{\text{Hausdorff}}{X}$ then K closed

Proof: 1. let $\{U_\alpha = \underbrace{V_\alpha}_{\text{open in } X} \cap K\}_{\alpha\in A}$ - covering of K. Then $\{V_\alpha\}_{\alpha\in A}, \underbrace{X\setminus K}_{\text{open}}$ - a cover of X

$\xrightarrow[X \text{ compact}]{}$ choose a finite subcover $\{V_\alpha\}_{\alpha\in A'}, X\setminus K$

$\Rightarrow \{U_\alpha\}_{\alpha\in A'}$ - a fin. cover of K

2. let $x\in X\setminus K$ want to prove that $\exists$ open nbhd of x in $X\setminus K$.



$\forall\, y\in K$ $\exists$ V_y open nbhd of y, U_y - open nbhd of x s.t. $V_y\cap U_y=\emptyset$
- by Hausdorff property.

$\{V_y\cap K\}_{y\in K}$ - open cover of K

$\xrightarrow{\text{choose a}}$ fin. subcover $\{V_{y_i}\cap K\}$ $\Rightarrow \bigcup_i V_{y_i}$ contains K but is disjoint from $\underset{\ni x}{\bigcap_i U_{y_i}}$ - open nbhd of x □

Proof of Prop*: need to prove that $g=f^{-1}: \underset{\text{Hausdorff}}{Y}\to \underset{\text{cpt}}{X}$ is continuous

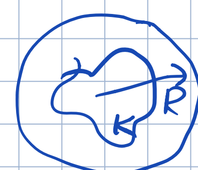
i.e. $\forall\, \underset{\text{open}}{U}\subset X$, $g^{-1}(U)=f(U)\subset Y$ open.

Passing to complements: need to prove that $f(\underset{\text{closed in } X}{C}) \underset{\text{closed!}}{\subset} Y$

X cpt $\underset{Lm^{\#\#},1}{\Rightarrow}$ C cpt $\underset{Lm^{\#}}{\Rightarrow}$ $f(C)$ cpt $\underset{Lm^{\#\#},2}{\Rightarrow}$ $f(C)$ closed. □

Heine-Borel Theorem

A subspace $K\subset\mathbb{R}^n$ is compact iff K is a closed subset of $\mathbb{R}^n$ and bounded, i.e. $\exists\, R>0$ s.t. $K\subset B_R(0)$



Ex: we had a cont. bijection

$$f: [-1,1]/\{\pm 1\} \longrightarrow S^1$$

domain: Heine-Borel $\Rightarrow [-1,1]$ cpt $\Rightarrow$ quotient $[-1,1]/\{\pm1\}$ cpt

codomain: S^1 - subspace of $\mathbb{R}^2 \underset{Lm^{@}}{\Rightarrow}$ Hausdorff

$\Bigg\}$ $\Rightarrow$ can use Prop* $\Rightarrow f$ - homeo!

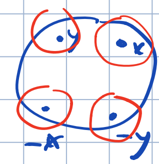
Ex we had a cont. bijection

$f: \quad D^n/_\sim \longrightarrow \mathbb{RP}^n$

domain - cpt. by HB + quotient

codomain: $\mathbb{RP}^n = S^n/_\sim$

let $x, y \in S^n$
$\left\{ x \neq -y \right.$
$[x] \neq [y] \in \mathbb{RP}^n$
$\{\pm x\} \quad \{\pm y\}$



r - small

$U = (B_r(x) \cup B_r(-x)) \cap S^n$

$V = (B_r(y) \cup B_r(-y)) \cap S^n$

$U \cap V = \emptyset$ (r small)

$p(U), p(V)$ - disjoint open nbhds of $[x]$, $[y]$ in $\mathbb{RP}^n$.

Stopped here

- A quotient of a Hausdorff space is not always Hausdorff!

<u>example:</u> $X = \mathbb{R} / (-1, 1)$

- points -1 and $+1$ do not have disjoint neighborhoods in X!

< Assume $\underset{-1}{\overset{\Psi}{U}}, \underset{1}{\overset{\Psi}{V}} \subset X$ disjoint open nbhds. $\leadsto p^{-1}U, p^{-1}V$ open nbhds of ± 1

$\rightarrow p^{-1}U$ overlaps with $(-1,1)$ $\Rightarrow$ contains it, and $p^{-1}V$ also

$\Rightarrow$ they are not disjoint - contradiction! >

<u>Proof of Heine-Borel:</u>

we need: fact 1: a closed interval $[a,b]$ is compact

fact 2: if $X_1, \dots, X_n$ are cpt top spaces, then $X_1 \times \dots \times X_n$ is cpt

<see Munkres>

- Let $K \underset{\text{cpt}}{\subset} \mathbb{R}^n \underset{\text{Lm \#\# 2}}{\Rightarrow} K$ closed.

$\{B_r(0) \cap K\}_{r>0}$ - open cover of K

$\underset{\text{compactness}}{\Rightarrow}$ choose a finite subcover $\{B_{r_i}(0) \cap K\}_{r_i}$

- take $R =$ the maximum radius r_i. Then $K \subset B_R(0)$

$\Rightarrow K$ bounded

- Conversely: let K - closed, bounded $\Rightarrow \underbrace{K}_{\text{closed}} \subset B_R(0) \subset \underbrace{[-R, R]^{\times n}}_{\text{cpt (facts 1,2)}} \Rightarrow$

$\underset{\text{Lm \#\# 1}}{\Rightarrow}$ K cpt $\boxed{/\!/}$

<u>Corollary of HB</u>: If $f: X \to \mathbb{R}$ is a cont fcn on a cpt space X, then f has a maximum and a minimum.

<u>Proof:</u> $K = f(X) \underset{\text{cpt}}{\subset} \mathbb{R} \Rightarrow$ it is bounded $\Rightarrow$ has $\inf = a$ and $\sup = b$. Those are limits of sequences in K. Since K is also closed, $a, b \in K$

$\Rightarrow f$ attains its minimum and maximum on X.