8/19/20 Quiz I

LAST TIME: • Heine-Borel THM: A subspace $K \subset \mathbb{R}^n$ is compact iff it is closed and bounded.

• Prop: $f: X \to Y$ a cont. bijection is a homeomorphism if X is compact and Y is Hausdorff.

• A quotient of a Hausdorff space is not always Hausdorff!

example: $X = \mathbb{R}/(-1,1)$ • points -1 and $+1$ do not have disjoint neighborhoods in X!

< Assume $\underset{-1}{\underset{\Psi}{U}}, \underset{1}{\underset{\Psi}{V}} \subset X$ disjoint open nbhds. $\rightsquigarrow p^{-1}U, p^{-1}V$ open nbhds of ± 1
$\to p^{-1}U$ overlaps with $(-1,1)$ & contains it, and $p^{-1}V$ also
$\Rightarrow$ they are not disjoint – contradiction! >

Proof of Heine-Borel:

we need: fact 1: a closed interval $[a,b]$ is compact
fact 2: if $X_1, \dots, X_n$ are cpt top spaces, then $X_1 \times \dots \times X_n$ is cpt
<see Munkres>

• Let $K \underset{cpt}{\subset} \mathbb{R}^n \underset{Lm \#2}{\Rightarrow} K$ closed.

$\{B_r(0) \cap K\}_{r>0}$ – open cover of K

$\underset{\text{compactness}}{\Rightarrow}$ choose a finite subcover $\{B_{r_i}(0) \cap K\}_{r_i}$
– take $R =$ the maximum radius r_i. Then $K \subset B_R(0)$

$\Rightarrow K$ bounded

• Conversely: let K - closed, bounded $\Rightarrow \underbrace{K}_{\text{closed}} \subset B_R(0) \subset \underbrace{[-R,R]^{\times n}}_{\text{cpt (facts 1,2)}} \Rightarrow$

$\underset{Lm \#1}{\Rightarrow} K$ cpt $\boxed{/\!/}$

Lemma (Corollary of HB): If $f: X \to \mathbb{R}$ is a cont fcn on a cpt space X, then f has a maximum and a minimum.

Proof: $K = f(X) \underset{cpt}{\subset} \mathbb{R} \Rightarrow$ it is bounded $\Rightarrow$ has $\inf = a$ and $\sup = b$. Those are limits of sequences in K. Since K is also closed, $a, b \in K$
$\Rightarrow f$ attains its minimum and maximum on X. $\exists x_{max}$ s.t. $f(x_{max}) = b \geq f(x)\ \forall x \in X$

Another proof: assume $K = f(X) \subset \mathbb{R}$ (cpt) does not have a maximum. Then it is covered by $K \subset \bigcup_{a \in K} (-\infty, a) \cap f(X) \xrightarrow{\text{choose a finite subcover}} K \subset \bigcup_i (-\infty, a_i) \cap f(X)$

→ if $a_1 = \max\{a_i\}$, then $a_i \notin K$ – contradiction! ⇒ K has a maximum
minimum – similar ☐

Connected spaces

def A top space X is connected if it cannot be written as $X = U \sqcup V$ with U, V non-empty disjoint open subsets of X.

Ex: let $a < b < c < d$
then
- $X = [a, b) \sqcup (c, d)$ is not connected
- $X' = [a, b] \sqcup [c, d]$ is not connected

Lemma Any interval $I \subset \mathbb{R}$ (open, closed, half-open; bounded or not) is connected

Proof: assume $I = U \sqcup V$ pick $u \in U, v \in V$ (assume $u < v$). Then $[u, v] = \underbrace{U'}_{[u,v] \cap U} \sqcup \underbrace{V'}_{[u,v] \cap V}$ – decomp into disj. union of non-empty open sets.

Claim: $c := \sup U'$ belongs to U' and V' – CONTRADICTION!

Indeed: • assume $c \notin U'$. $\forall \varepsilon > 0$ can find an elt u_i in $(c - \varepsilon, c) \rightsquigarrow$ seq. $\{c_i\}$ converging to c
$\overset{\cap}{U'}$ ⇒ $c \in U'$ [shorter: $\sup U' =$ limit point of $U' \in U'$ since U' closed]

$U' \subset [u, v]$ closed (its complement V' is open)

• $\forall x \in [u, v]$, $x > c = \sup U'$, is in V' ⇒ can construct a seq. $v_i \in V_i$ converging to c $\underset{V' \text{ closed } \subset [u,v]}{\Longrightarrow}$ $c \in V'$! [or: ⇒ $c = \inf V' \in V'$] ☐

Intermediate value theorem: Let X be a connected top. space and $f: X \to \mathbb{R}$ a cont. map. If $a, b \in f(X)$, then $\forall c$ s.t. $a < c < b$ is in $f(X)$.

Proof: Assume $c \notin f(X)$. Then $X = f^{-1}(-\infty, c) \sqcup f^{-1}(c, \infty)$ – a decomposition of X as a union of non-empty disjoint open sets ☐

def: A top space X is path connected if $\forall x, y \in X$ there is a path connecting x and y, i.e. $\exists\, \gamma : \underbrace{[a, b]}_{\text{some interval}} \xrightarrow{\text{cont.}} X$ with $\gamma(a) = x$, $\gamma(b) = y$.

Lemma Any path connected top space X is connected.

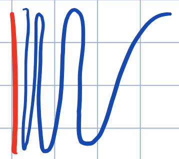
Proof: Assume X is path conn. but not connected ⇝ $X = U \sqcup V$.
X path conn. ⇒ $\exists\, \gamma : [a, b] \to X$, $\gamma(a) \in U$, $\gamma(b) \in V$ ⇒ $[a, b] = \underset{\ni a}{\gamma^{-1}(U)} \sqcup \underset{\ni b}{\gamma^{-1}(V)}$
⇒ $[a, b]$ not connected – contradiction! ☐

stopped here

- For "nice" top. spaces, path connected $\Leftrightarrow$ connected.

Counterexample - "topologist's sine curve" - connected but not path connected

$$X = \left\{ (x, \sin \tfrac{1}{x}) \in \mathbb{R}^2 \mid 0 < x < 1 \right\} \cup \{ (0, y) \in \mathbb{R}^2 \mid -1 \le y \le 1 \} \qquad \subset \mathbb{R}^2$$



- CW complexes

"closure-finite"
"weak topology"

Def A CW complex (= "cell complex") is a Hausdorff topological space X and a collection of subsets $e_\alpha \subset X$ (cells) with the following properties:

(i) $X = \bigcup_\alpha e_\alpha$, $\quad e_\alpha \cap e_\beta = \emptyset$ for $\alpha \neq \beta$

(ii) Each cell is equipped with a characteristic map $\varphi_\alpha : D^k \longrightarrow X$, where D^k - closed k-dim. disk ($k \geq 0$ the dimension of the cell), s.t.

- $\varphi_\alpha|_{\mathrm{int}(D^k)}$ is a homeo of the interior of the disk onto $e_\alpha \subset X$
- the boundary ∂D^k is sent to $(k-1)$-skeleton of X,
  $\varphi_\alpha(\partial D^k) \subset X_{k-1}$, where $X_k := \bigcup_{\alpha,\ \dim e_\alpha \leq k} e_\alpha$.

automatic for finite CW complexes

(iii) Each closure $\underset{\varphi_\alpha(D^k)}{\overline{e_\alpha}}$ is contained in the union of finitely many cells

(iv) a set $Y \subset X$ is closed in X iff $Y \cap \overline{e_\alpha}$ is closed in $\overline{e_\alpha}$ for all e_α.

The map $\varphi_\alpha|_{\partial D^k}$ is called the attaching map for the cell e_α.

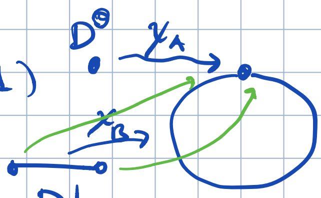
Skeleta form an increasing filtration of X: $\quad X_0 \subset X_1 \subset \cdots \subset X$, $X = \bigcup_k \mathrm{sk}_k X$

- a map $f : X \to Y$ is called cellular if $f(X_k) \subset Y_k \ \forall k$
- a subcollection of cells $Y = \bigcup_\beta e_\beta$, $\{\beta\} \subset \{\alpha\}$ is a "subcomplex" of X, if $\overline{e_\beta} \subset Y \ \forall e_\beta$

· One builds a CW complex inductively: X_0 = collection of points (0-cells)

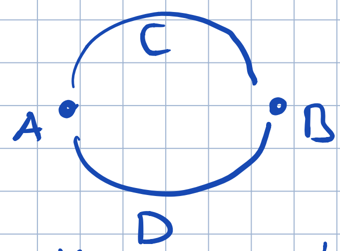
$$X_k = X_{k-1} \sqcup D^k_{(1)} \sqcup \cdots \sqcup D^k_{(N_k)} \Big/ \underset{1 \le \alpha \le N_k}{x \in \partial D^k_{(\alpha)}} \sim \varphi_\alpha(x) \in X_{k-1}$$

__Examples:__ (1)



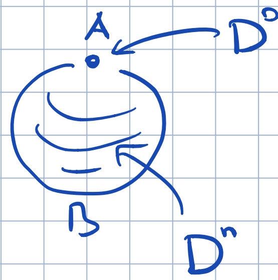
$X \approx S^1$

(2)



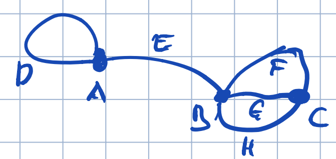
X also $\approx S^1$

(3)



$\approx S^n$

(4) graph = 1-dimensional CW complex



(5) $\mathbb{RP}^n = D^n \cup \mathbb{RP}^{n-1}$

$\varphi : S^n \xrightarrow{2-1} \mathbb{RP}^{n-1}$, $\quad \partial D^n$

$\Rightarrow$ by induction, $\mathbb{RP}^n$ has a CW structure with a single cell in each dimension $0, 1, \dots, n$

$\mathbb{RP}^n = e^0 \cup e^1 \cup \dots \cup e^n$