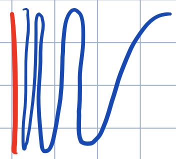
- For "nice" top. spaces, path connected $\Longleftrightarrow$ connected.

Counterexample – "topologist's sine curve" – connected but not path connected

$$X = \left\{ \left(x, \sin\frac{1}{x}\right) \in \mathbb{R}^2 \mid 0 < x < 1 \right\} \cup \left\{ (0,y) \in \mathbb{R}^2 \mid -1 \le y \le 1 \right\} \subset \mathbb{R}^2$$



- CW complexes

"closure-finite"
"weak topology"

Def A CW complex (= "cell complex") is a Hausdorff topological space X and a collection of subsets $e_\alpha \subset X$ (cells) with the following properties:

(i) $X = \bigcup_\alpha e_\alpha$, $e_\alpha \cap e_\beta = \varnothing$ for $\alpha \neq \beta$

(ii) Each cell is equipped with a characteristic map $\varphi_\alpha : D^k \longrightarrow X$, where D^k – closed k-dim. disk ($k \geq 0$ the dimension of the cell), s.t.

- $\varphi_\alpha|_{\text{int}(D^k)}$ is a homeo of the interior of the disk onto $e_\alpha \subset X$
- the boundary ∂D^k is sent to $(k-1)$-skeleton of X, $\varphi_\alpha(\partial D^k) \subset X_{k-1}$, where $X_k := \bigcup_{\alpha,\ \dim e_\alpha \leq k} e_\alpha$.

automatic for finite CW complexes

(iii) Each closure $\overline{e_\alpha}$ $\left(= \varphi_\alpha(D^k)\right)$ is contained in the union of finitely many cells

(iv) a set $Y \subset X$ is closed in X iff $Y \cap \overline{e_\alpha}$ is closed in $\overline{e_\alpha}$ for all e_α.

The map $\varphi_\alpha|_{\partial D^k}$ is called the attaching map for the cell e_α.

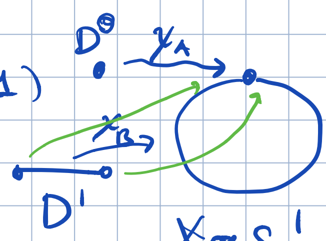
Skeleta form an increasing filtration of X: $X_0 \subset X_1 \subset \cdots \subset X$, $X = \bigcup_k \text{sk}_k X$

- a map $f: X \to Y$ is called cellular if $f(X_k) \subset Y_k$ $\forall k$
- a subcollection of cells $Y = \bigcup_\beta e_\beta$, $\{\beta\} \subset \{\alpha\}$ is a "subcomplex" of X, if $\overline{e_\beta} \subset Y$ $\forall e_\beta$

· One builds a CW complex inductively: X_0 = collection of points (0-cells)

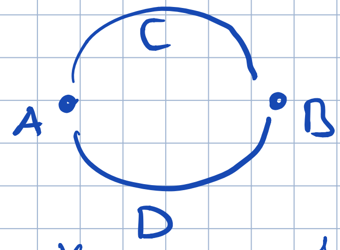
$$X_k = X_{k-1} \amalg D^k_{(1)} \amalg \cdots \amalg D^k_{(N_k)} \Big/ {\scriptstyle x \in \partial D^k_{(\alpha)} \sim \varphi_\alpha(x) \in X_{k-1},\ 1 \le \alpha \le N_k}$$

Examples: (1)



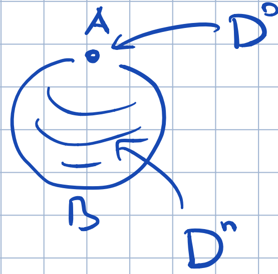
$X \approx S^1$

(1')



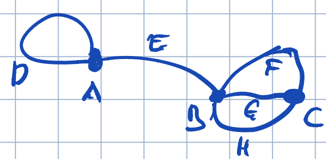
X also $\approx S^1$

(2)



$\approx S^n$

(3) graph = 1-dimensional CW complex



(4) $\mathbb{RP}^n = D^n \cup \mathbb{RP}^{n-1}$ $\quad \varphi: \underset{\partial D^n}{\overset{\shortparallel}{S^n}} \xrightarrow{2:1} \mathbb{RP}^{n-1}$

$\Rightarrow$ by induction, $\mathbb{RP}^n$ has a CW structure with a single cell in each dimension $0, 1, \dots, n$

$\mathbb{RP}^n = e^0 \cup e^1 \cup \dots \cup e^n$

Topological manifolds

def A manifold of dimension n (or "n-manifold") is a top. space X which is locally homeomorphic to $\mathbb{R}^n$, i.e. $\forall x \in X$ has an open nbhd $U \underset{x}{\overset{\cup}{\subset}} X$ s.t. $U \underset{\text{homeo}}{\approx} V \underset{\text{open}}{\subset} \mathbb{R}^n$. Additionally, X is assumed to be Hausdorff and second countable (i.e. $\exists$ countable basis of topology).

Rem: If X is Hausdorff then any subspace $Y \subset X$ is Hausdorff.
If X is 2^{nd} countable, then —— " —— is 2^{nd} countable

<Homework>

Ex: 1. Any open $U \subset \mathbb{R}^n$ is an n-manifold.
(Note: $\mathbb{R}^n$ is 2^{nd} countable — countable basis: $\{B_r(x) \mid r \in \mathbb{Q}_{>0},\ x \in \mathbb{Q}^n\}$

Stopped here

3

2. $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ is an n-manifold

- covered by open subsets $U_i^+ = \{x = (x_0, \dots, x_n) \mid x_i > 0\} \subset S^n$ $\quad i = 0 \dots n$

$U_i^- = \{ \text{ —— '' —— } \mid x_i < 0\} \subset S^n$

$\varphi_i^\pm : U_i^\pm \longrightarrow \overset{\circ}{D}{}^n = \{(v_1, \dots, v_n) \mid v_1^2 + \dots + v_n^2 < 1\} \subset \mathbb{R}^n$ — homeomorphism

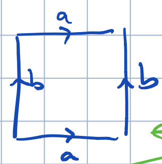
open disk

$x \longmapsto (x_0, \dots, \hat{x}_i, \dots, x_n)$

$(v_1, \dots, \pm\sqrt{1 - \|v\|^2}, \dots, v_n) \longleftarrow\!\shortmid (v_1, \dots, v_n)$

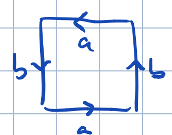
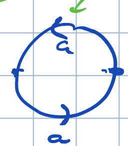
- If X, Y are manifolds of dim. m, n, then $X \times Y$ is an $(m+n)$-manifold (homework)
- $\mathbb{RP}^n$ is an n-manifold. (homework)

Examples of 2-manifolds.

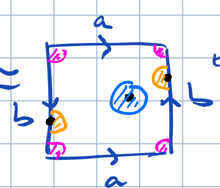
(1) 2-torus $T \approx$  $\approx S^1 \times S^1$ — product of 1-mfds $\Rightarrow$ 2-mfd

$T \subset \mathbb{R}^3$

edge identifications

(2) $\mathbb{RP}^2 \approx$  $\approx$ 

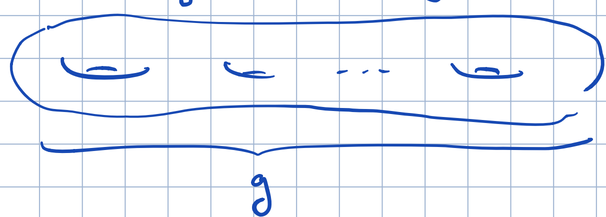
"bigon" - 2 vertices, 2 edges

(3) Klein bottle $K \approx$  ← nbhds of - bulk pt - edge pt - vertex

We'll also see later that $K \approx \mathbb{RP}^2 \# \mathbb{RP}^2 \Rightarrow$ manifold

connected sum

(4) surface Σ_g of genus g - subspace of $\mathbb{R}^3$ given by the picture



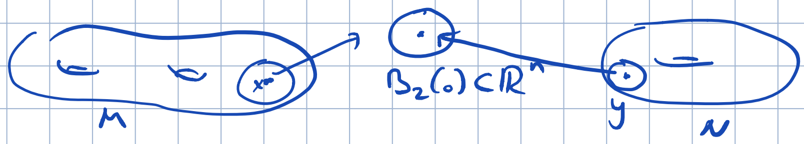
$\Sigma_1 = T$ torus, $\Sigma_0 = S^2$ sphere

Connected sum construction

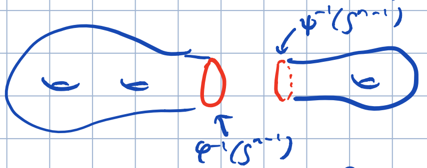
M, N - n-mfds $\longrightarrow M \# N$ - new n-mfd - "connected sum"

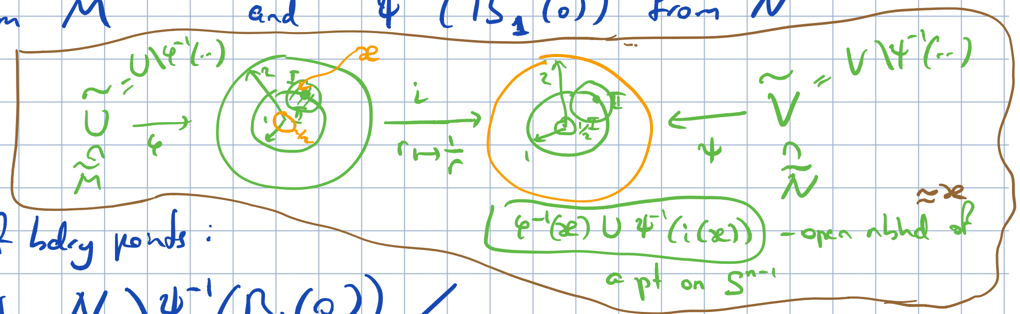
- choose $x \in M$, $y \in N$
- pick a homeo $\varphi : U \xrightarrow{\simeq} B_2(0) \subset \mathbb{R}^n$ and $\psi : V \xrightarrow{\simeq} B_2(0)$

U: open nbhd of x



- remove $\varphi^{-1}(B_1(0))$ from M and $\psi^{-1}(B_1(0))$ from N





- Pass to the identification of bdry points:

$$M\#N := M \setminus \varphi^{-1}(B_1(0)) \cup N \setminus \psi^{-1}(B_1(0)) \Big/ \varphi^{-1}(S^{n-1}) \xrightarrow{\approx} \psi^{-1}(S^{n-1})$$
$$z \longmapsto \psi^{-1}(\varphi(z))$$

Fact: - (assuming M, N connected) $M\#N$ up to homeo is independent of the choices (of φ, ψ) "if one is careful with orientations"

- for 2-manifolds, $M\#N$ is always indep. of choices.

Ex: (1) from pictures: $\Sigma_2 \# T \approx \Sigma_3$, $\Sigma_g \# \Sigma_{g'} \approx \Sigma_{g+g'}$.

$\Rightarrow \underbrace{T\#T\# \cdots \#T}_{g} \approx \Sigma_g$ (we will view this as a definition of Σ_g)

(2) $X_k = \underbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}_{k}$ - "k-fold projective plane" (in Munkres' terminology)

Theorem (Classification of compact connected 2-manifolds)
Every compact connected 2-manifold is homeo to exactly one of the following manifolds:

- the genus g surface, $g \geq 0$: $\Sigma_g = \underbrace{T\# \cdots \#T}_{g>0}$ or $\Sigma_0 = S^2$
- $X_k = \underbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}_{k \geq 1}$

There are two aspects here: (1) 2-mfds $\Sigma_0, \Sigma_1, \Sigma_2, \ldots, X_1, X_2$ are pairwise non-homeomorphic

(2) any cpt conn 2-mfd Σ is homeo to a mfd on the list

(1) $\Leftarrow$ Euler characteristic + orientability

<u>Rem</u> One has:

- $\mathbb{RP}^2 \# \mathbb{RP}^2 \approx K$ – Klein bottle
- $\mathbb{RP}^2 \# T \approx \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$