Ex: $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ is an n-manifold

- covered by open subsets $U_i^+ = \{x = (x_0, \dots, x_n) \mid x_i > 0\} \subset S^n$, $U_i^- = \{ \quad \text{''} \quad \mid x_i < 0\} \subset S^n$, $i = 0 \dots n$

$$\varphi_i^\pm : U_i^\pm \longrightarrow \overset{\circ}{D}{}^n = \{(v_1, \dots, v_n) \mid v_1^2 + \dots + v_n^2 < 1\} \subset \mathbb{R}^n \quad \text{– homeomorphism}$$

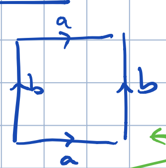
open disk

$$x \longmapsto (x_0, \dots, \hat{x}_i, \dots, x_n)$$

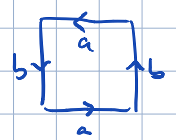
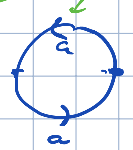
$$(v_1, \dots, \pm\sqrt{1 - \|v\|^2}, \dots, v_n) \longleftarrow (v_1, \dots, v_n)$$

- If X, Y are manifolds of dim. m, n, then $X \times Y$ is an $(m+n)$-manifold (homework)
- $\mathbb{RP}^n$ is an n-manifold. (homework)

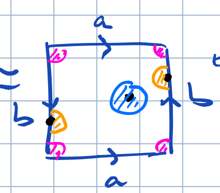
Examples of 2-manifolds.

(1) 2-torus $T \underset{\mathbb{R}^3}{\cap} \approx$  $\approx S^1 \times S^1$ – product of 1-mfds $\Rightarrow$ 2-mfd

edge identifications

(2) $\mathbb{RP}^2 \approx$  $\approx$ 

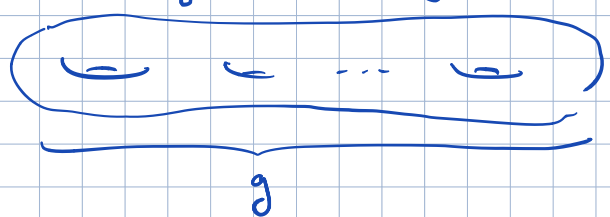
"bigon" – 2 vertices, 2 edges

(3) Klein bottle $K \approx$  ← nbhds of
- bulk pt
- edge pt
- vertex

We'll also see later that $K \approx \mathbb{RP}^2 \# \mathbb{RP}^2 \Rightarrow$ manifold

connected sum

(4) surface Σ_g of genus g – subspace of $\mathbb{R}^3$ given by the picture



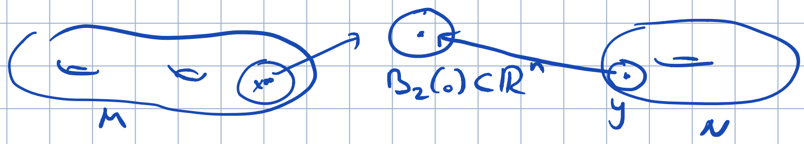
$\Sigma_1 = T$ torus, $\Sigma_0 = S^2$ sphere

Connected sum construction

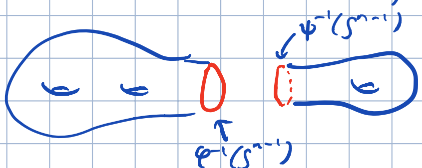
M, N – n-mfds $\longrightarrow M \# N$ – new n-mfd – "connected sum"

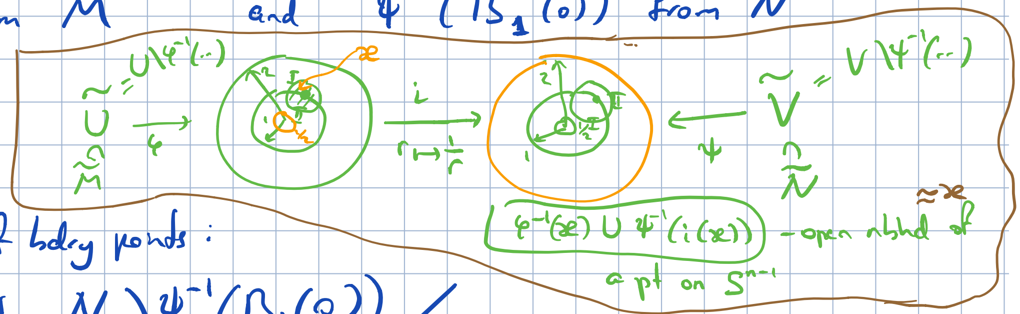
- choose $x \in M$, $y \in N$
- pick a homeo $\varphi : U \xrightarrow{\approx} B_2(0) \subset \mathbb{R}^n$ and $\psi : V \xrightarrow{\approx} B_2(0)$

open nbhd of x



- remove $\varphi^{-1}(B_1(0))$ from M and $\psi^{-1}(B_1(0))$ from N





- Pass to the identification of bdry points:

$$M\#N := M \setminus \varphi^{-1}(B_1(0)) \cup N \setminus \psi^{-1}(B_1(0)) \Big/ \varphi^{-1}(S^{n-1}) \xrightarrow{\approx} \psi^{-1}(S^{n-1})$$

$$z \longmapsto \psi^{-1}(\varphi(z))$$

Fact: (assuming M, N connected)
- $M\#N$ up to homeo is independent of the choices (of φ, ψ) "if one is careful with orientations"
- for 2-manifolds, $M\#N$ is always indep. of choices.

Ex: (1) from pictures: $\Sigma_2 \# T \approx \Sigma_3$, $\Sigma_g \# \Sigma_{g'} \approx \Sigma_{g+g'}$.

$\Rightarrow \underbrace{T\#T\# \cdots \#T}_{g} \approx \Sigma_g$ (we will view this as a <u>definition</u> of Σ_g)

(2) $X_k = \underbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}_{k}$ - "k-fold projective plane" (in Munkres' terminology)

Theorem (Classification of compact connected 2-manifolds)

Every compact connected 2-manifold is homeo to exactly one of the following manifolds:

- the genus g surface, $g \geq 0$: $\Sigma_g = \underbrace{T\# \cdots \# T}_{g>0}$ or $\Sigma_0 = S^2$
- $X_k = \underbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}_{k \geq 1}$

There are two aspects here: (1) 2-mfds $\Sigma_0, \Sigma_1, \Sigma_2, \dots, X_1, X_2$ are pairwise non-homeomorphic

(2) any cpt conn 2-mfd Σ is homeo to a mfd on the list

(1) $\Leftarrow$ Euler characteristic + orientability

Rem One has:

- $\mathbb{RP}^2 \# \mathbb{RP}^2 \approx K$ – Klein bottle
- $\mathbb{RP}^2 \# T \approx \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$

Euler characteristic

In S. Stolz's notes – "pattern of polygons"

Let Σ - compact 2-manifold. A cell decomposition (CW) γ of Σ = a graph Γ on Σ:

a collection of fin. many points (vertices) $v_1, \ldots, v_k \in \Sigma$,

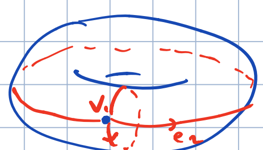
— " ——— paths (edges) $e_1, \ldots, e_\ell : [0,1] \to \Sigma$

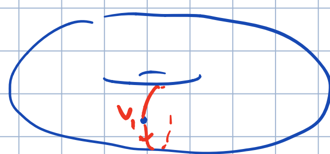
s.t. - endpoints belong to $V = \{v_1, \ldots, v_k\}$

- intersections of paths occur only at the endpoints

such that $\Sigma \setminus \Gamma \approx \coprod_i \mathring{D}^2$

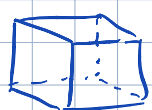
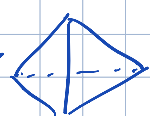
Ex:





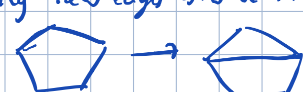
- Euler characteristic: $\chi(\Sigma, \gamma) := \#\text{vertices} - \#\text{edges} + \#\text{polygons}$
  (0-cells) (1-cells) (2-cells)

Ex:

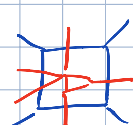
 $\approx S^2 \approx$ 

$8 - 12 + 6 = 2$ $\qquad$ $4 - 6 + 4 = 2$

Lemma: Let γ, γ' be two cell decompositions of a cpt 2-mfd Σ.
Then $\chi(\Sigma, \gamma) = \chi(\Sigma, \gamma')$

Argument:
- by moving Γ, Γ' a bit, can make their vertex sets disjoint and have fin. many intersections between edges of Γ and Γ'
- can construct γ'' - a common refinement of γ, γ' (i.e. γ'' is obtained from γ, resp γ', by inductively adding new vertices on edges  or adding new edges btw vertices of a polygon 

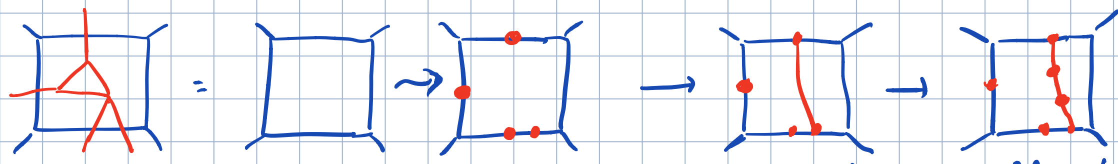
how? $V_{\gamma''} = V_\gamma \amalg V_{\gamma'} \amalg (\Gamma \cap \Gamma')$
intersection pts of edges of Γ and Γ'



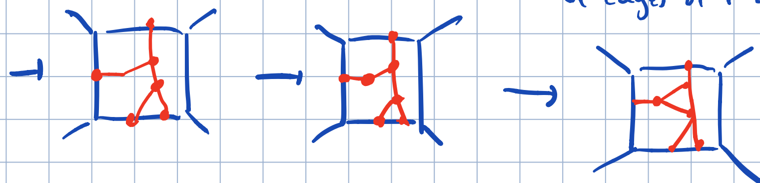
edges of γ'' = segments of edges of γ, γ' connecting vertices of γ''

Claim: γ'' is indeed a refinement of γ, and of γ'.

Idea: (working locally in a polygon of γ)



add intersection pts of edges of Γ with Γ' — add edge connecting them — add vertices on that edge



(*) If γ_2 is obtained from γ_1 by adding a vertex on an edge

then $\chi(\Sigma,\gamma_2) = V_{\Gamma_2} - E_{\Gamma_2} + F_{\Gamma_2} = (V_{\Gamma_1} + \cancel{1}) - (E_{\Gamma_1} + \cancel{1}) + F_{\Gamma_1} = \chi(\Sigma,\gamma_1)$

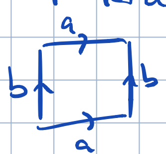
(**) If γ_2 is obtained from γ_1 by splitting a polygon by an edge

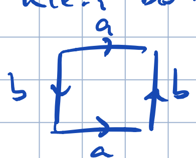
then $\chi(\Sigma,\gamma_2) = V_{\Gamma_2} - E_{\Gamma_2} + F_{\Gamma_2} = V_{\Gamma_1} - (E_{\Gamma_1} + 1) + (F_{\Gamma_1} + 1)$
$= \chi(\Sigma,\gamma_1)$

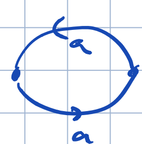
Thus (*, **) $\Rightarrow \chi(\Sigma,\gamma) = \chi(\Sigma,\underset{\substack{\uparrow \\ \text{refinement}}}{\gamma''}) = \chi(\Sigma,\gamma')$

def: Let Σ be a compact 2-manifold. The Euler characteristic of Σ is defined to be the integer $\chi(\Sigma) = \chi(\Sigma,\gamma)$ ← any cell decomposition.

Ex: T = torus — Klein bottle — $\mathbb{RP}^2$







$\chi = 1 - 2 + 1 = 0$ — $\chi = 1 - 2 + 1 = 0$ — $\chi = 1 - 1 + 1 = 1$

- A homeo $f: \Sigma \cong \Sigma'$ maps a cell decomp of Σ to cell decomp of Σ'

$\Rightarrow$ Euler char of homeo mflds agrees.

Corollary: S^2, T and $\mathbb{RP}^2$ are pairwise non-homeomorphic

Lemma: For Σ, Σ' cpt 2-mflds, $\chi(\Sigma \# \Sigma') = \chi(\Sigma) + \chi(\Sigma') - 2$.
(homework)

Corollary: $\chi(\underbrace{\Sigma_g}_{T\#\cdots\#T}) = 2-2g$, $\chi(\underbrace{X_k}_{\mathbb{RP}^2\#\cdots\#\mathbb{RP}^2}) = 2-k$. In particular $\Sigma_g \simeq \Sigma_{g'}$ iff $g = g'$ and $X_k \simeq X_{k'}$ iff $k = k'$

g

k