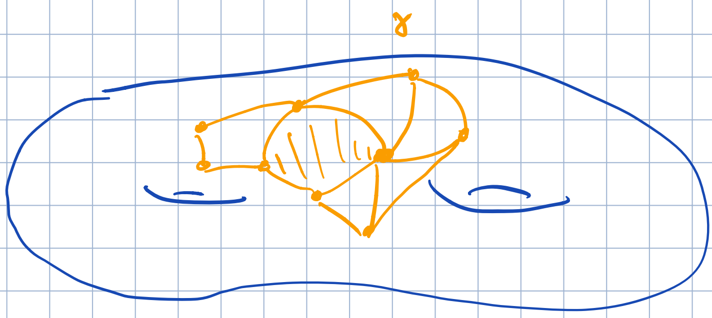
LAST TIME

Σ = compact connected 2-manifold with γ - cell decomposition ("pattern of polygons")
Γ - corresponding graph on Σ



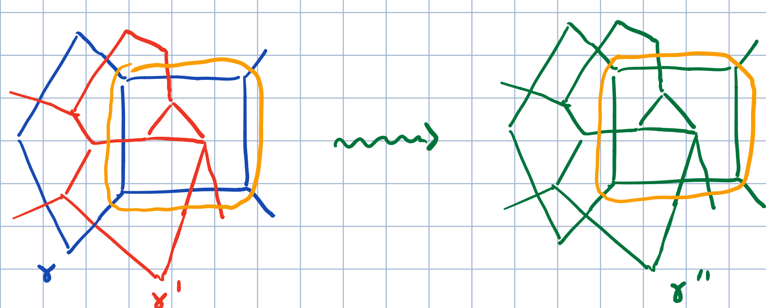
Σ

$$\chi(\Sigma,\gamma) := \#\text{ vertices } - \#\text{ edges } + \#\text{ polygons}$$
(0-cells) (1-cells) (2-cells)

Lemma: $\chi(\Sigma,\gamma) = \chi(\Sigma,\gamma')$ for γ,γ' two cell decomp. of Σ.

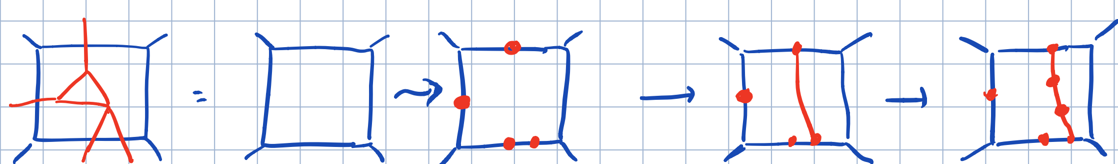
Argument
- moving Γ, Γ' can make V_Γ disj from $V_{\Gamma'}$ and edges of Γ have fin. many intersections with edges of Γ'
- $\gamma \rightsquigarrow \gamma''$, $\gamma' \rightsquigarrow \gamma''$ "common refinement"

γ''

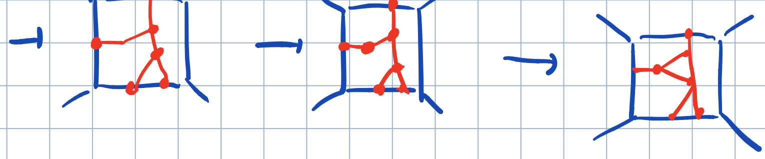


$V_{\Gamma''} = V_\Gamma \amalg V_{\Gamma'} \amalg$ intersections of edges of Γ and Γ'

$\gamma \rightsquigarrow \gamma''$



add intersection pts of edges of Γ with Γ' — add edge connecting them — add vertices on that edge

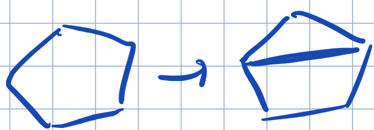


* If $\gamma_1 \rightsquigarrow \gamma_2$

$$\chi(\Sigma,\gamma_1) = \chi(\Sigma,\gamma_2)$$

$$\left(\begin{array}{l} \#V_{\gamma_1} - \#E_{\gamma_1} + \#F_{\gamma_1} \\ \underbrace{\#V_{\gamma_2}}_{(\#V_{\gamma_1}+1)} - \underbrace{\#E_{\gamma_2}}_{(\#E_{\gamma_1}+1)} + \underbrace{\#F_{\gamma_2}}_{\#F_{\gamma_1}} \end{array}\right.$$

* If $\gamma_1 \to \gamma_2$



$$\chi(\Sigma,\gamma_1) = \chi(\Sigma,\gamma_2)$$

$$\underset{+0}{\#V} - \underset{+1}{\#E} + \underset{+1}{\#F}.$$

$$\begin{array}{l} \gamma \searrow \\ \qquad \gamma'' \\ \gamma' \nearrow \end{array} \qquad \chi(\Sigma,\gamma) = \chi(\Sigma,\gamma'') = \chi(\Sigma,\gamma')$$

$\blacksquare$

def

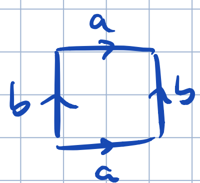
Let Σ be a cpt conn 2-mfd

The Euler char of Σ is: $\chi(\Sigma) = \chi(\Sigma, \gamma)$

any cell decomp.

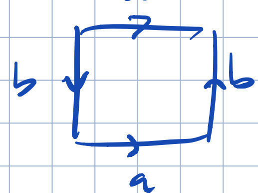
Ex:

T = torus



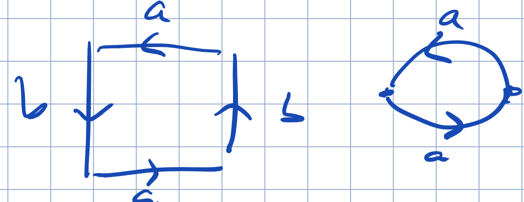
$\chi = 1 - 2 + 1 = 0$

Klein bottle



$\chi = 1 - 2 + 1 = 0$

$\mathbb{RP}^2$



$\chi = 1 - 1 + 1$
$= 1$

A homeo $\Sigma \approx \Sigma'$ maps a cell decomp of Σ

onto a cell decomp of Σ

$\Rightarrow$ χ of homeomorphic manifolds agrees.
(Euler char.)

<u>Corollary</u>: S^2, T, RP^2 are not homeomorphic!

<u>Lemma</u>: for Σ, Σ' cpt conn 2-mfds

$$\boxed{\chi(\Sigma \# \Sigma') = \chi(\Sigma) + \chi(\Sigma') - 2}$$

<u>Corollary</u>

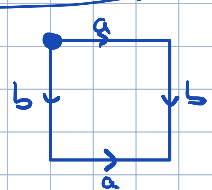
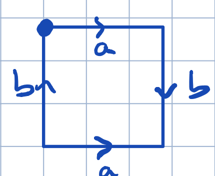
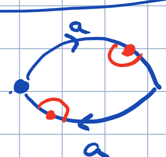
$$\chi(\underbrace{\Sigma_g}_{T\#\cdots\#T}) = 2-2g \quad , \quad \chi(\underbrace{X_k}_{\mathbb{RP}^2\#\cdots\#\mathbb{RP}^2}) = 2-k$$

In part., $\Sigma_g \approx \Sigma_{g'}$ iff $g=g'$
$X_k \approx X_{k'}$ iff $k=k'$.

A combinatorial description of cpt conn 2-mfds

2 mfds which can be obtained by gluing edges of a single polygon $\longleftrightarrow$ "words"

<u>Ex</u>:

2-mfd	edge gluing	word
torus		$aba^{-1}b^{-1}$
Klein bottle		$aba^{-1}b$
$\mathbb{RP}^2$		aa

... letters $\in A = \{a, a^{-1}, b, b^{-1}, \dots\}$

surf obtained by gluing edges of an n-gon

(edges $\to$ labels + orientation)

seq. of letters ... labels $\{a,b,\dots\} = L$ - set of labels

$\rightsquigarrow$ word: fix a vertex, go clockwise

$W = x_1 \dots x_n$

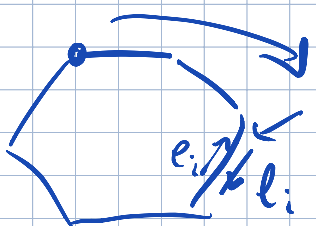
$x_i = (\ell_i)^{\pm 1}$ $\Leftarrow$ depends on orientation of the edge

ℓ_i - label of e_i

reverse:

$W = \ell_1^{\pm 1} \ell_2^{\pm 1} \dots \ell_n^{\pm 1} \longmapsto P_n / \sim_W =: \Sigma(W)$

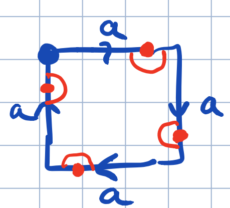
n-gon

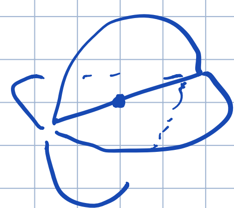


e_i ℓ_i depending on ± 1 in W

Rem: $\Sigma(W)$ is not always a manifold

$\Sigma(a\,a\,a\,a)$





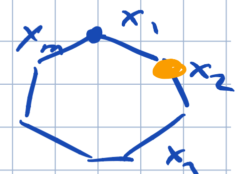
Lemma A: (1) W - word built from labels in L

$L \overset{i}{\longleftrightarrow} L'$ bijection

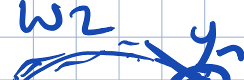
$W' = i(W)$

- $\Sigma(W') \approx \Sigma(W)$

(2) $\Sigma(x_1 \dots x_m) \approx \Sigma(x_2 \dots x_m x_1)$



more generally $\Sigma(\underbrace{W_1}_{x_1 \dots x_m} \underbrace{W_2}_{y_1 \dots y_n}) \approx \Sigma(W_2 W_1)$



y1
x1
xm
W1