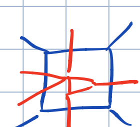
Lemma: Let γ, γ' be two cell decompositions of a cpt 2-mfd Σ.
Then $\chi(\Sigma, \gamma) = \chi(\Sigma, \gamma')$

Argument:
- by moving Γ, Γ' a bit, can make their vertex sets disjoint and have fin. many intersections between edges of Γ and Γ'
- can construct γ'' - a common refinement of γ, γ' (i.e. γ'' is obtained from γ, resp γ', by inductively adding new vertices on edges  or adding new edges btw vertices of a polygon

how? $V_{\gamma''} = V_{\gamma} \amalg V_{\gamma'} \amalg (\Gamma \cap \Gamma')$

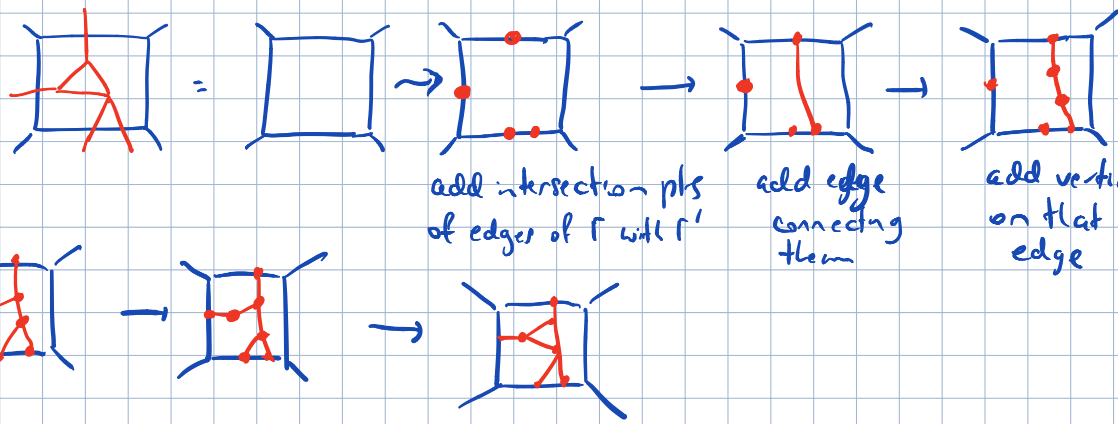
intersection pts of edges of Γ and Γ'

edges of γ'' = segments of edges of γ, γ' connecting vertices of γ''



Claim: γ'' is indeed a refinement of γ, and of γ'. (1)

Idea: (working locally in a polygon of γ)



(*) If γ_2 is obtained from γ_1 by adding a vertex on an edge

then $\chi(\Sigma, \gamma_2) = V_{\Gamma_2} - E_{\Gamma_2} + F_{\Gamma_2} = (V_{\Gamma_1} \cancel{+1}) - (E_{\Gamma_1} \cancel{+1}) + F_{\Gamma_1} = \chi(\Sigma, \gamma_1)$

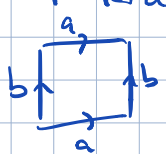
(**) If γ_2 is obtained from γ_1 by splitting a polygon by an edge

then $\chi(\Sigma,\gamma_2) = V_{\Gamma_2} - E_{\Gamma_2} + F_{\Gamma_2} = V_{\Gamma_1} - (E_{\Gamma_1} + 1) + (F_{\Gamma_1} + 1)$
$= \chi(\Sigma, \gamma_1)$

Thus $(*,**) \Rightarrow \chi(\Sigma,\gamma) = \chi(\Sigma, \underset{\substack{\uparrow \\ \text{refinement}}}{\gamma''}) = \chi(\Sigma, \gamma')$

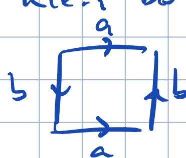
def: Let Σ be a compact 2-manifold. The Euler characteristic of Σ is defined to be the integer $\chi(\Sigma) = \chi(\Sigma, \gamma)$ ← any cell decomposition.

Ex: T = torus



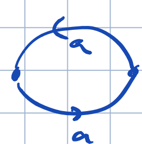
$\chi = 1 - 2 + 1 = 0$

Klein bottle



$\chi = 1 - 2 + 1 = 0$

$\mathbb{RP}^2$



$\chi = 1 - 1 + 1 = 1$

- A homeo $f : \Sigma \simeq \Sigma'$ maps a cell decomp of Σ to cell decomp of Σ'

$\Rightarrow$ Euler char of homeo mflds agrees.

Corollary: S^2, T and $\mathbb{RP}^2$ are pairwise non-homeomorphic

Lemma: For Σ, Σ' cpt 2-mflds, $\chi(\Sigma \# \Sigma') = \chi(\Sigma) + \chi(\Sigma') - 2$.

(homework)

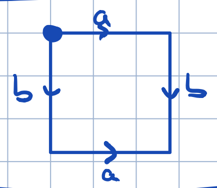
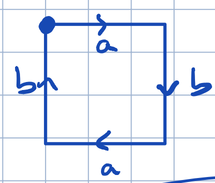
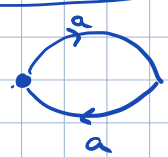
Corollary: $\chi(\underbrace{\Sigma_g}_{T\#\cdots\#T}) = 2 - 2g$, $\chi(\underbrace{X_k}_{\mathbb{RP}^2\#\cdots\#\mathbb{RP}^2}) = 2 - k$. In particular $\Sigma_g \simeq \Sigma_{g'}$ iff $g = g'$ and $X_k \simeq X_{k'}$ iff $k = k'$

g k

A combinatorial description of compact connected 2-manifolds

2-mfds obtained by gluing edges of a single polygon $\longleftrightarrow$ "words"

Ex:

2-mfd	edge gluing	word
torus		$aba^{-1}b^{-1}$
Klein bottle		$aba^{-1}b$
$\mathbb{RP}^2$		aa

surface given by edge-gluing of an n-gon (edges → labels) + orientation $\longmapsto$ word : fix a vertex and go clockwise from it

(seq. of "letters" $\in A = \{a, a^{-1}, b, b^{-1}, \dots\}$; $a, b, \dots \in L$ - labels)

alphabet

set of labels

$W = x_1 x_2 \cdots x_n$

$x_i = (\ell_i)^{\pm 1}$ — label of i^{th} edge e_i

power ± 1 : $+1$ if e_i is oriented clockwise, -1 if counterclockwise

reverse direction:

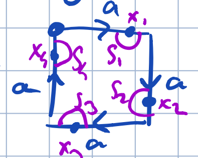
word $W = \ell_1^{\pm 1} \cdots \ell_n^{\pm 1}$

$$\longmapsto P_n / \sim_W =: \Sigma(W)$$

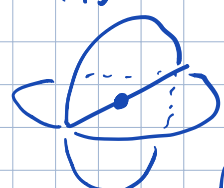
P_n: n-gon with a distinguished vertex

$\sim_W$: mark i^{th} edge e_i (going clockwise) by label ℓ_i, orient clockwise if ℓ_i^{+1}, —"— counterclockwise if ℓ_i^{-1} appears in W.

Rem $\Sigma(W)$ is not always a manifold.

Ex: $\Sigma(aaaa) =$ 

open nbhd of $[x_1] = S_1 \cup \dots \cup S_4 / \sim_W$



- gluing of 4 semi-disks.
- not a manifold!

Lemma A 1) Let W be a word built from labels in a set L. Let $L \leftrightarrow L'$ be a bijection of sets and let W' be the word obtained by replacing each occurrence of $\ell^{\pm 1}$ with $\ell'^{\pm 1}$ where ℓ' corresponds to ℓ via bijection. Then

$$\boxed{\Sigma(W) \approx \Sigma(W')}.$$

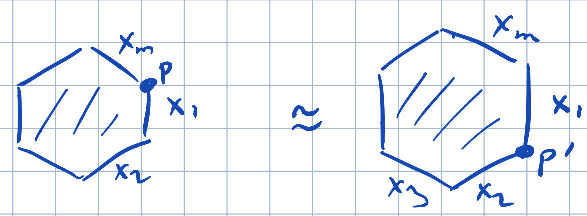
(2) $$\boxed{\Sigma(x_1 x_2 \cdots x_m) \approx \Sigma(x_2 x_3 \cdots x_m x_1)}$$

more generally: for $\underset{x_1 \cdots x_m}{W_1}, \underset{y_1 \cdots y_n}{W_2}$ two words with letters in A,

(@) $$\boxed{\Sigma(\underbrace{W_1 W_2}_{x_1 \cdots x_m y_1 \cdots y_n}) \approx \Sigma(\underbrace{W_2 W_1}_{y_1 \cdots y_n x_1 \cdots x_m})}$$

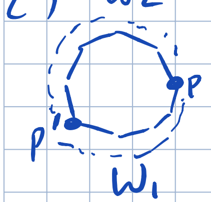
concatenation of words

Proof: (1): obvious (2)



- same edge identification $\Rightarrow$ same Σ, just start labelling edges from a different vertex.

(2')



- can start labelling from p or from p'. $\square$

Proposition Let M, N be two cpt, conn 2-mfds, $M = \Sigma(W_1)$, $N = \Sigma(W_2)$, W_1, W_2 - words from <u>disjoint alphabets</u>. Then the connected sum is

$$M \# N = \Sigma(W_1 W_2)$$

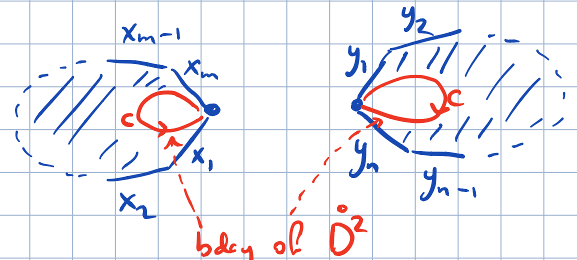
Proof $W_1 = x_1 \cdots x_m$ $W_2 = y_1 \cdots y_n$

$M = \Sigma(W_1) =$

$= \Sigma(W_2) = N$

remove $\mathring{D}^2$ from M and N

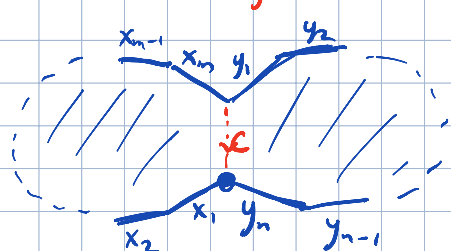
$M \setminus \mathring{D}^2 =$



$= N \setminus \mathring{D}^2$

glue along the circle c

$M \# N =$



$\Rightarrow M \# N = \Sigma(x_1 \cdots x_m y_1 \cdots y_n) = \Sigma(W_1 W_2)$ $\boxed{/\!/}$

Corollary: (1) $\Sigma_g = \underbrace{T \# \cdots \# T}_{g} \approx \Sigma(a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1})$

$\approx \Sigma(a_1 b_1 a_1^{-1} b_1^{-1}) \# \cdots \# \Sigma(a_g b_g a_g^{-1} b_g^{-1})$

(2) $X_k = \underbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}_{k} \approx \Sigma(a_1 a_1 a_2 a_2 \cdots a_k a_k)$

$\approx \Sigma(a_1 a_1) \# \cdots \# \Sigma(a_k a_k)$

Proposition B: Let W_1, W_2, W_3 be words and a a letter not occurring in them.
Then there are homeomorphisms

(*) $\Sigma(W_1 \, a W_2 a W_3) \approx \Sigma(W_1 \, aa W_2^{-1} W_3)$,

(**) $\Sigma(W_1 \, a W_2 a \, W_3) \approx \Sigma(W_1 W_2^{-1} aa W_3)$

where W_2^{-1} is the "inverse" of the word W_2: $W_2 = x_1 \cdots x_n \to W_2^{-1} = x_n^{-1} \cdots x_1^{-1}$.
(as for a group product)

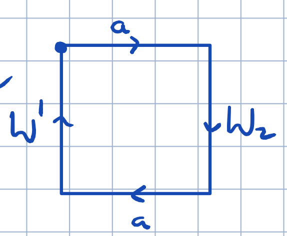
Proof: <let's check (*)>

$\Sigma(W_1 a W_2 a W_3) \underset{(@)}{\approx} \Sigma(a W_2 a \underbrace{W'}_{W_3 W_1})$

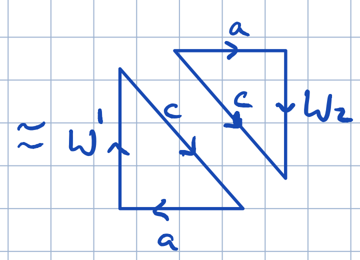
$\Sigma(W_1 aa W_2^{-1} W_3) \approx \Sigma(aa W_2^{-1} W')$

So, we want to prove: $\Sigma(a W_2 a W') \approx \Sigma(aa W_2^{-1} W')$

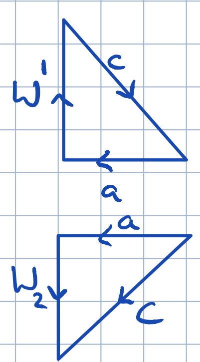
$\Sigma(a W_2 a W') \approx$



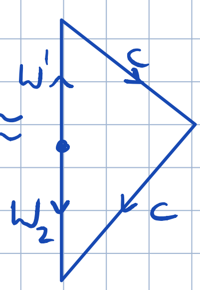
$\approx$



$\approx$



$\approx$



$\approx \Sigma(cc W_2^{-1} W') \approx \Sigma(aa W_2^{-1} W')$

$\boxed{/\!/}$

(**) is similar - we cut the square by the other diagonal

Application: proof of $T \# \mathbb{RP}^2 \approx \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$

Indeed: $T \# \mathbb{RP}^2 = \Sigma(aba^{-1}b^{-1}) \# \Sigma(cc) \approx \Sigma(aba^{-1}b^{-1}cc)$

$\approx \Sigma(abcbac) \approx \Sigma(abbc^{-1}ac) \approx \Sigma(bbc^{-1}aca)$

$\approx \Sigma(bbc^{-1}c^{-1}aa) \approx \Sigma(bb) \# \Sigma(c^{-1}c^{-1}) \# \Sigma(aa) \approx \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$

$\square$

- Idea of proof of "constructive part" of the classification THM
  <that any cpt conn 2-mfd is $\approx \Sigma_g$ or X_k>

(1) show that every Σ admits a triangulation (cell decomp where polygons = triangles)
→ give each edge its own label

$\Rightarrow \Sigma = \coprod \text{polygons} \big/$ gluing edges carrying same label

(2) Reduce the number of polygons by one by gluing a pair of edges with same label belonging to different polygons

⟶ inductively, reduce to a single polygon

(3) Use the moves of Lemma A, Prop. B to show that labeling of edges of the polygon can be modified, without changing the homeo type of the resulting quotient, to obtain the stand. labeling for Σ_g or X_k.