REMINDER: Classification THM: $\underset{\text{cpt conn 2-mfd}}{\underset{\uparrow}{\Sigma}} \approx \Sigma_g$ or X_k for some $g \geq 0$ or $k \geq 1$.

Combinatorial description: $\Sigma = \Sigma(\underset{\text{word}}{\underset{\uparrow}{W}})$

Lemma A: • relabelling: if $L \xrightarrow[1-1]{i} L'$ bijection on the set of labels,
then $\Sigma(i(W)) \approx \Sigma(W)$

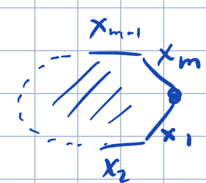
• moving the initial vertex: $\Sigma(W_1 W_2) \approx \Sigma(W_2 W_1)$
(cyclicity)

Proposition B Let M, N be two cpt conn 2-mflds
$M = \Sigma(W_1)$ $N = \Sigma(W_2)$ W_1, W_2 words from disjoint alphabets

Then $\boxed{M \# N = \Sigma(W_1 W_2)}$.

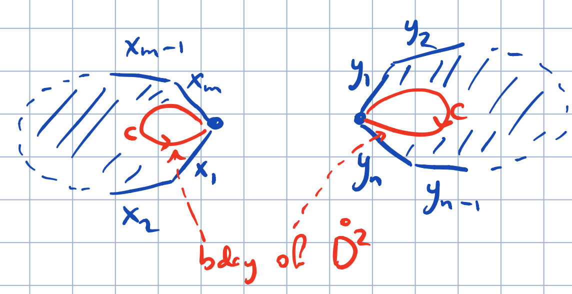
Argument

$W_1 = x_1 \cdots x_m$ $W_2 = y_1 \cdots y_n$

$M = \Sigma(W_1) =$ 

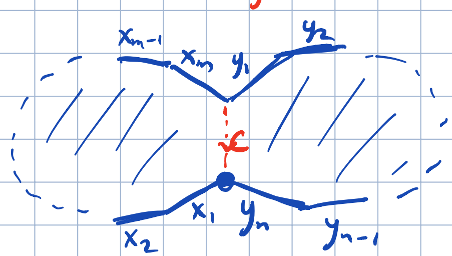
$= \Sigma(W_2) = N$

remove $\mathring{D}^2$ from M and N

$M \setminus \mathring{D}^2 =$ 

$= N \setminus \mathring{D}^2$

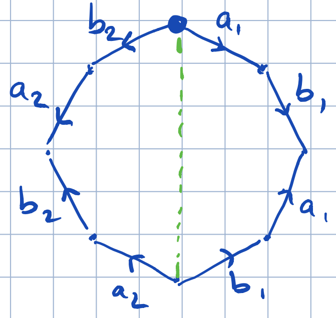
glue along the circle c

$M \# N =$ 

$\Rightarrow M \# N = \Sigma(x_1 \cdots x_m y_1 \cdots y_n) = \Sigma(W_1 W_2)$ ☐

Corollary (1) $\Sigma_g = \underbrace{T \# \cdots \# T}_{g} = \Sigma(a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1})$

$= \Sigma(a_1 b_1 a_1^{-1} b_1^{-1}) \# \cdots \# \Sigma(a_g b_g a_g^{-1} b_g^{-1})$



Σ_2

(2) $X_k = \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2 = \Sigma(a_1 a_1 \cdots a_k a_k)$

$= \Sigma(a_1 a_1) \# \cdots \# \Sigma(a_k a_k)$

Proposition C: Let W_1, W_2, W_3 - words, a - letter not occurring in any of them

Then there are the following homeos

(*) $\Sigma(W_1 a W_2 a W_3) \approx \Sigma(W_1 a a W_2^{-1} W_3)$

$W_2 = x_1 \cdots x_n \qquad W_2^{-1} = x_n^{-1} \cdots x_1^{-1}$

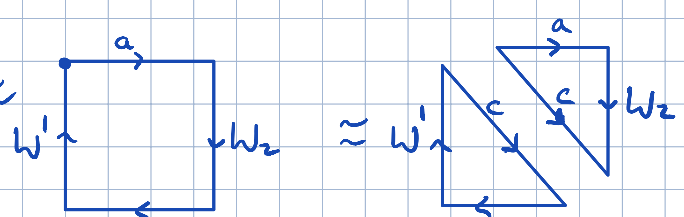
(**) $\Sigma(W_1 a W_2 a W_3) \approx \Sigma(W_1 W_2^{-1} a a W_3)$

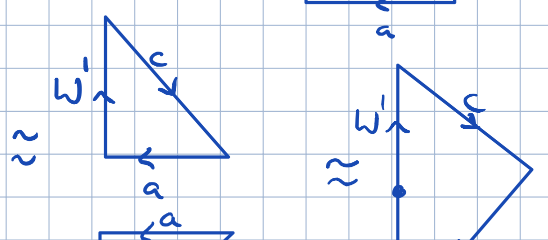
Proof (*) $\Sigma(W_1 a W_2 a W_3) \underset{A}{\approx} \Sigma(a W_2 a \underbrace{W_3 W_1}_{W'})$

$\Sigma(W_1 a a W_2^{-1} W_3) \underset{A}{\approx} \Sigma(a a W_2^{-1} \underbrace{W_3 W_1}_{W'})$

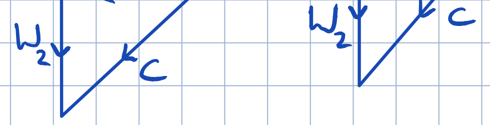
want to prove $\boxed{\Sigma(a W_2 a W') \overset{!}{\approx} \Sigma(a a W_2^{-1} W')}$ $\forall\, W_2, W'$

$\Sigma(a W_2 a W') \approx$





$\approx \Sigma(c c W_2^{-1} W') \approx \Sigma(a a W_2^{-1} W')$



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Application: $\underbrace{T \# \mathbb{RP}^2}$ $\approx \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$

$$\text{lhs} = \Sigma(aba^{-1}b^{-1}) \# \Sigma(cc) \approx \Sigma(\overbrace{ab}^{w_1}\overbrace{a^{-1}b^{-1}}^{w_2}cc) \approx \Sigma(abcbac) \approx$$

Prop C

$$\approx \underbrace{\Sigma(abbc^{-1}ac)}_{A} \approx \Sigma(bbc^{-1}aca) = \Sigma(\underline{bb}\,\underline{c^{-1}c^{-1}}\,\underline{aa})$$

$$\approx \Sigma(bb) \# \Sigma(c^{-1}c^{-1}) \# \Sigma(aa)$$

$$= \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$$

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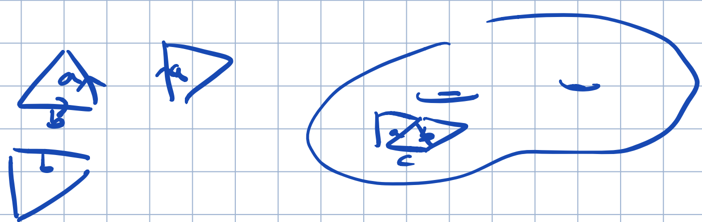
Idea of proof of "constructive part" of the classification THM

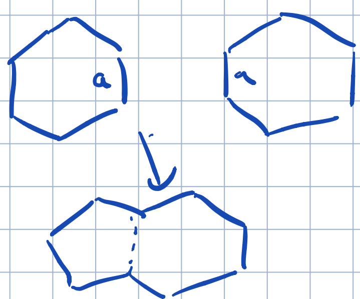
(1) show that every cpt con. 2-mfd admits a cell decomp. (pattern of polygons)

(in fact, a triangulation exists)

$\Sigma = \coprod \text{polygon}/\sim$



(2) reduce the number of polygons inductively



$\rightsquigarrow$ eventually, Σ is given by edge-gluing of a single polygon

(3) Use Lemma A, Prop C to show that labelling of edges of the polygon can be modified (without changing homeo type of the quotient) to the stand. labelling of Σ_g or X_k.

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def A 2-mfd Σ is called non-orientable if it contains a subspace homeo to the Möbius band. Otherwise, Σ is orientable.

Proposition: (i) X_k is non-orientable

(ii) Σ_g is orientable.

Rem: if $\Sigma \overset{f}{\cong} \Sigma'$ a homeo then both are orientable or not.

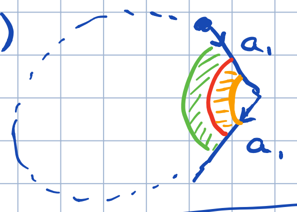
Thus Prop $\Rightarrow$ $X_k \not\cong \Sigma_g$ for any g and k

Σ_2 $\quad X_3$ - not homeo!

$\chi = -2 \quad \chi = -2$

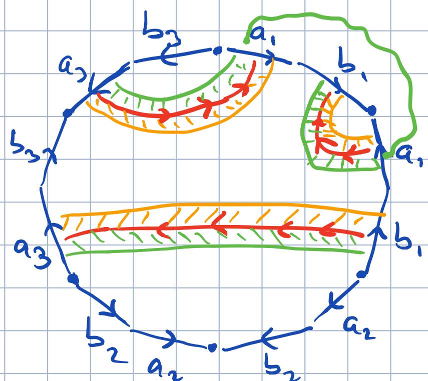
Proof:

(i)

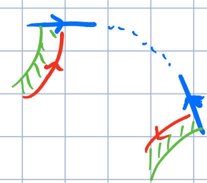


(ii) Σ_g

$\therefore$ $\underset{\text{Möbius}}{M} \hookrightarrow \Sigma_g$ - homeo onto its image



4g-gon



$\boxtimes$