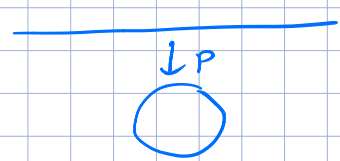
Covering spaces

def A map $p: \tilde{X} \to X$ is a covering map if $\forall x \in X$ $\exists$ an open nbhd $\underset{x}{\overset{\psi}{U}} \subset X$ s.t. $p^{-1}(U) = \coprod_i U_i \subset \tilde{X}$ s.t. $p|_{U_i}: U_i \to U$ is a homeomorphism for each U_i.

Such U is called evenly covered. $\tilde{X}$ is called a covering space for X.

Ex: 1) $\underset{\tilde{X}}{\mathbb{R}} \xrightarrow{p} \underset{X}{S^1}$
$t \longmapsto e^{2\pi i t}$



2) $S^1 \xrightarrow{p} S^1$ - each point has n preimages $\to$ "n-sheeted" cover
$z \longmapsto z^n$

3) $S^n \xrightarrow{p} \mathbb{RP}^n = S^n/\sim$ - 2-sheeted cover
quotient map

Prop[a] <"lemma (a)"> (Unique path lifting for covering spaces)
Let $\gamma: I \to X$ a path starting at $x_0 \in X$ and let $\tilde{x}_0 \in p^{-1}(x_0)$.
Then $\exists!$ path $\tilde{\gamma}: I \to \tilde{X}$ s.t.
- $p \circ \tilde{\gamma} = \gamma$ and
- $\tilde{\gamma}(0) = \tilde{x}_0$.

- $\tilde{\gamma}$ is called the "lift" of γ.

(Proof: as lemma (a))

Prop[c] <"lemma (c)"> (Unique homotopy lifting)
Let $p: \tilde{X} \to X$ a covering space, $f_t: Y \to X$ a homotopy (of maps from Y) and $\tilde{f}_0: Y \to \tilde{X}$ a map lifting f_0. Then $\exists!$ homotopy $\tilde{f}_t$ starting with $\tilde{f}_0$ that lifts f_t

(Proof: as lemma (c))

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "\tilde{f}_0"] \arrow[d, "i"'] & \tilde{X} \arrow[d, "p"] \\ Y \times I \arrow[r, "f"'] \arrow[ur, dashed, "\exists! \tilde{f}"] & X \end{tikzcd}$$

- Case $Y = I$: f_0 = path in X, $\tilde{f}_0$ = path in $\tilde{X}$ (lift)

if f_t - homotopy of paths in X $\Rightarrow$ $\tilde{f}_t$ - homotopy of paths in $\tilde{X}$

Observation: homotopy $\tilde{f}_t$ preserves endpoints iff f_t preserves endpoint

Γ $\Rightarrow$) obvious: $\underset{\text{or } 1}{f_t(0)} = p(\underset{\text{or } 1}{\tilde{f}_t(0)})$ - constant

$\Leftarrow$ $\underset{\text{or } 1}{f_t(0)}$, $t \in [0,1]$ - constant path in X $\longrightarrow$ $\tilde{f}_t(0)$, $t \in [0,1]$ some path in $\tilde{X}$, but must be constant by uniqueness of path lifting $\tilde{f}_t(0) = \tilde{f}_0(0)$ $\lrcorner$

Proposition:

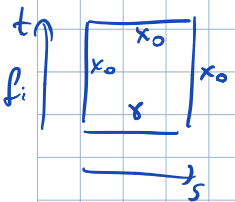
Let $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ be a covering map. Then

(i) The induced homomorphism $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_1(X, x_0)$ is injective.

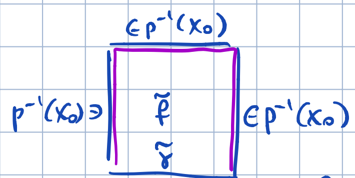
(ii) A based loop γ in (X, x_0) represents an element of the image of p_* iff its (unique) lift $\tilde{\gamma} : I \longrightarrow \tilde{X}$ with $\tilde{\gamma}(0) = \tilde{x}_0$ is a loop, i.e., $\tilde{\gamma}(1) = \tilde{x}_0$.

Proof: Use homotopy lifting property, for $Y = I$.

(i) let $\tilde{\gamma}$ be a based loop in $(\tilde{X}, \tilde{x}_0)$ in $\ker p_*$. Let $f : \underset{s}{I} \times \underset{t}{I} \longrightarrow X$ be a homotopy from $\overset{\gamma}{p \circ \tilde{\gamma}}$ to C_{x_0} (const loop)



Using Prop (c): let $\tilde{f}$ be a lift of f with $f|_{t=0} = \tilde{\gamma}$



by uniqueness of path lifting, $\tilde{f}$ on left-top-right sides is the const. path $C_{\tilde{x}_0}$.

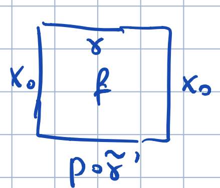
~~$\Rightarrow \tilde{\gamma}(1) = \tilde{\gamma}(0) = x_0 \Rightarrow \tilde{\gamma}$ a loop based at $\tilde{x}_0$.~~

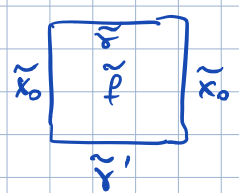
$\tilde{f}$ is a homotopy between $\tilde{\gamma}$ and $C_{\tilde{x}_0}$

$\Rightarrow [\tilde{\gamma}] = 0$.

(ii) let $\underset{\substack{\uparrow \\ \text{based loop in } X, x_0}}{[\gamma]} \in \operatorname{im} p_*$. I.e. $\exists\, \underset{\substack{\text{based} \\ \text{Loop in } \tilde{X}, \tilde{x}_0}}{\tilde{\gamma}'}$ s.t. $[\gamma] = p_*[\tilde{\gamma}'] = [p \circ \tilde{\gamma}']$

let f · homotopy between γ and $p \circ \tilde{\gamma}'$, and $\tilde{f}$ its lift with $f|_{t=0} = \tilde{\gamma}'$.





$\Rightarrow$ the lift $\tilde{\gamma}$ of γ starting at $\tilde{x}_0$ also ends at $\tilde{x}_0$.

$\Rightarrow \tilde{\gamma}$ is a based loop. ▨

· for $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ covering map, introduce the map

$$W : \{\text{based loops in } (X, x_0)\} \longrightarrow p^{-1}\{x_0\}$$
$$\gamma \longmapsto \tilde{\gamma}(1)$$

where $\tilde{\gamma}$ is the lifting of γ with $\tilde{\gamma}(0) = \tilde{x}_0$.

(W generalizes the idea of a winding number $W : \{\text{loops in } S^1\} \to \mathbb{Z}$ in the case of $p : \mathbb{R} \to S^1$)

<u>Proposition</u> Let $\tilde{X}$ be path-connected and let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering map. Let $G = \pi_1(X, x_0)$ and $H := \underbrace{p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subset G}_{\text{subgroup}}$.

Let $H \backslash G = \{Hg \mid g \in G\} = G \big/ \substack{g \sim hg \\ \forall g \in G, h \in H}$

– the set of left H-cosets.

Then the map
$$\begin{aligned} \Psi : H \backslash G &\longrightarrow p^{-1}(x_0) \\ [\gamma] &\longmapsto W(\gamma) \end{aligned}$$
is a well-defined bijection.

In particular, the number of sheets of $\tilde{X} \to X$ (the cardinality of $p^{-1}(x_0)$) is equal to the index $\underbrace{[G : H]}_{:= \# H \backslash G}$ of the subgroup $H \subset G$.

<u>Proof</u>

- <u>well-definedness</u>:

$$\underset{[\gamma]}{\overset{}{g}} \overset{\Psi}{\longmapsto} W(\gamma)$$

$$\underset{[\delta]\cdot[\gamma]}{hg} \overset{\Psi}{\longmapsto} W(\tilde{\gamma}) = \Psi(g) \quad \checkmark$$

$$[\delta]\cdot[\gamma] = [p \circ (\tilde{\delta} * \tilde{\gamma})]$$

lifts to a loop $\tilde{\delta}$ — lifts to a path $\tilde{\gamma}$ — starts at $\tilde{x}_0$, ends at $W(\gamma)$

- <u>Surjectivity</u>: (use that $\tilde{X}$ is path connected). For $\tilde{x}_1 \in p^{-1}(x_0)$, choose any path $\tilde{\gamma}$ from $\tilde{x}_0$ to $\tilde{x}_1$. Then $\Psi(\underbrace{[p \circ \tilde{\gamma}]}_{\text{loop in } X}) = \tilde{\gamma}(1) = \tilde{x}_1$. $\checkmark$

- <u>injectivity</u>: let $\underset{[\gamma_1]}{g_1}, \underset{[\gamma_2]}{g_2} \in \pi_1(X, x_0)$ with $\underset{W(\gamma_1)}{\Psi(g_1)} = \underset{W(\gamma_2)}{\Psi(g_2)} =: \tilde{x}_1$

$\tilde{\gamma}_1 * \overline{\tilde{\gamma}_2}$ – based loop in $(\tilde{X}, \tilde{x}_0)$ $\Rightarrow$ $H \ni [p(\tilde{\gamma}_1 * \overline{\tilde{\gamma}_2})] = g_1 g_2^{-1}$

from $\tilde{x}_0$ to $\tilde{x}_1$ — from $\tilde{x}_1$ to $\tilde{x}_0$

$\Rightarrow g_1 = h g_2$ for some $h \in H$ $\Rightarrow$ left H-cosets $Hg_1 = Hg_2$. $\checkmark$

$\square$

Lifting maps Given a covering $\begin{matrix}(\tilde X, \tilde x_0)\\ \downarrow p \\ (X, x_0)\end{matrix}$ and a map $(Y, y_0) \xrightarrow{f} (X, x_0)$,

we are interested in a *lifting*

$$\begin{tikzcd} & (\tilde X, \tilde x_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, dashed, "\tilde f"] \arrow[r, "f"'] & (X, x_0) \end{tikzcd} \qquad (*)$$

necessary condition for existence of $\tilde f$: $f_* \pi_1(Y, y_0) \subset p_* \pi_1(\tilde X, x_0)$, since we have

$$\begin{tikzcd} & \pi_1(\tilde X, \tilde x_0) \arrow[d, "p_* \text{ - inclusion!}"] \\ \pi_1(Y, y_0) \arrow[ur, dashed, "\tilde f_*"] \arrow[r, "f_*"'] & \pi_1(X, x_0) \end{tikzcd}$$

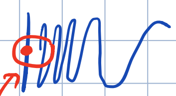
This is also a sufficient condition if Y is not "too wild".

def A top space Y is "locally path-connected" if $\forall y \in Y$ and any nbhd U of y, there is an open nbhd $V \subset U$ which is path-connected.

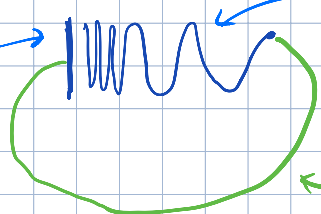
More generally, if P is some property of a top space (e.g. compact, connected, ...), then Y is "locally P" if $\forall y \in Y$, $\forall \underset{y}{\overset{\cup}{U}}$ nbhd, $\exists \underset{\text{open}}{V \subset U}$ s.t. V has property P.

Ex:

- $Y = Y_1 \amalg Y_2$ is locally path connected but not path connected
  (Y_1, Y_2 path connected manifolds)

- topologist's sine curve  connected but not locally path connected.
  these points don't have connected nbhds

- the "Warsaw circle"



graph of $\sin\frac{1}{x}$ on $x \in (0, a)$

$\{0\} \times [-1, 1]$

$\leftarrow$ circle arc connecting $(0,0)$ and $(a, \sin\frac{1}{a})$

- path connected but not locally path connected

Proposition (lifting criterion) Let $p: (\tilde X, \tilde x_0) \to (X, x_0)$ be a covering map and $f: (Y, y_0) \to (X, x_0)$ a (basepoint preserving) map whose domain Y is path connected and locally path connected. Then a lift $\tilde f$ in $(*)$ exists iff $f_* \pi_1(Y, y_0) \subset p_* \pi_1(\tilde X, \tilde x_0)$. There is at most one such lift.

Rem The hypothesis that Y is LPC cannot be dropped.

Ex: Y = Warsaw circle, $\pi_1(Y) = 0$

$f: Y \to S^1$ wrapping Y around S^1 once - this map does not have a lift $\tilde{f}: Y \to \mathbb{R}$.

Proof of lifting criterion: $\Rightarrow$ (lift $\exists \Rightarrow f_*\pi_1(Y) \subset \pi_1(\tilde{X})$) - already proved.

uniqueness: let $\tilde{f}, \tilde{f}'$ two lifts, $y \in Y$ and γ a path from y_0 to y ($I \to Y$)

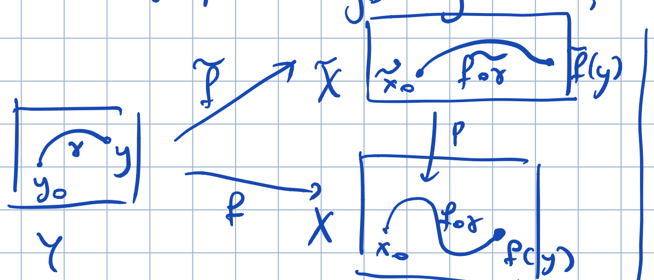
then $\tilde{f}\circ\gamma$, $\tilde{f}'\circ\gamma$ two paths in $\tilde{X}$ starting at $\tilde{x}_0$ and projecting to $f\circ\gamma$ in X

$\Rightarrow$ by uniqueness of lifted paths, $\underset{\tilde{f}(y)}{\underset{\shortparallel}{\tilde{f}\circ\gamma(1)}} = \underset{\tilde{f}'(y)}{\underset{\shortparallel}{\tilde{f}'\circ\gamma(1)}}$. This is so for any $y \in Y$ $\Rightarrow \tilde{f} = \tilde{f}'$ ✓

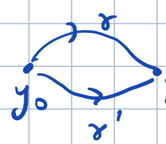
$\Leftarrow$ (Construction of the lift $\tilde{f}: Y \to \tilde{X}$)

$\tilde{f}: y \longmapsto \widetilde{f\circ\gamma}(1)$

where γ - path from y_0 to y in Y, $\widetilde{f\circ\gamma}$ - lift of $f\circ\gamma$ to a path in $\tilde{X}$ starting at $\tilde{x}_0$



• $\tilde{f}$ is well-defined: if



two paths,

$\gamma * \bar{\gamma}'$ - based loop in Y

$\Rightarrow f\circ(\gamma * \bar{\gamma}')$ - based loop in X

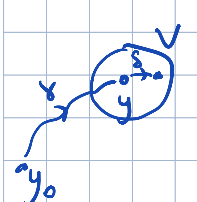
$[f\circ(\gamma * \bar{\gamma}')] = [p\circ\tilde{\delta}]$ — $\tilde{\delta}$ based loop in $\tilde{X}$

$f_*\pi_1(Y, y_0) \subset p_*\pi_1(\tilde{X}, \tilde{x}_0)$

by uniqueness of lifted paths:

$\underset{\text{based loop}}{\underset{\uparrow}{\tilde{\delta}}} = \widetilde{f\circ\gamma} * \widetilde{f\circ\bar{\gamma}'}$ $\Rightarrow$ $t=1$ endpoints of $\widetilde{f\circ\gamma}$ and $\widetilde{f\circ\gamma'}$ coincide! ✓

Continuity of $\tilde{f}$ - check in a nbhd of $y \in Y$. Use LPC: take $\underset{y}{\underset{\cup}{V}} \subset Y$ open PC nbhd of $f^{-1}(\underset{\text{evenly covered in } X}{\underset{\uparrow}{U}})$ (subset)



$\tilde{f}(y') = \widetilde{f(\gamma * \delta)}(1) = p_i^{-1} f(y')$ - depends continuously on y'! ✓

$y' \in V$

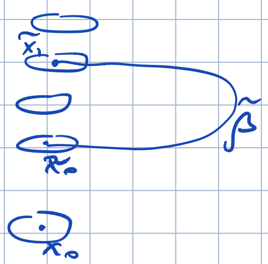
$p_i: \underset{\text{contains } \tilde{f}(y)}{\underset{\uparrow}{\tilde{U}_i}} \xrightarrow{\cong} U$

$\boxtimes$

Lemma Let $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ be a covering map and $\tilde{x}_1\in p^{-1}(x_0)$.
Let $\tilde{\beta}$ be a path from $\tilde{x}_0$ to $\tilde{x}_1$ in $\tilde{X}$ and $b=[p\circ\tilde{\beta}]\in\pi_1(X,x_0)$.
Let $H=p_*\pi_1(\tilde{X},\tilde{x}_0)$. Then

$$p_*\pi_1(\tilde{X},\tilde{x}_1)=b^{-1}Hb\subset\pi_1(X,x_0)$$

- conjugate subgroup.



Proof:

$$\begin{array}{ccc}
\Phi_\beta:\pi_1(\tilde{X},\tilde{x}_0) & \longrightarrow & \pi_1(\tilde{X},\tilde{x}_1) \\
[\tilde{\gamma}] & \longmapsto & [\bar{\tilde{\beta}}*\tilde{\gamma}*\tilde{\beta}] \\
 & & \downarrow p_* \\
 & & \pi_1(X,x_0) \\
 & & b^{-1}\cdot[p(\tilde{\gamma})]\cdot b
\end{array}$$

- isomorphism (change of base point)

$\square$