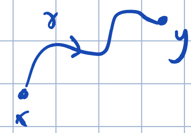
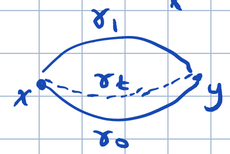
LAST TIME

Notes on the course webpage

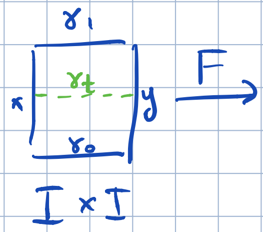
- paths $\gamma : [0,1] \to X$



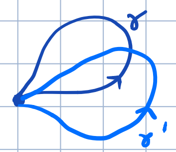
- homotopy of paths $\gamma_0 \sim \gamma_1$



or



- fundamental group $\pi_1(X, x_0) = \dfrac{\text{closed paths } x_0 \to x_0 \quad (\text{"based loops"})}{\text{homotopy}}$



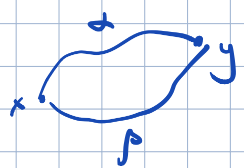
not assoc. for paths

assoc. for homotopy classes

concatenation of paths $\alpha * \beta \rightsquigarrow$ multiplication in π_1

reversal of a path $\bar{\alpha} \rightsquigarrow$ inverse in π_1

constant path at $x_0 \rightsquigarrow$ unit in π_1



α, β paths in $\underset{\text{convex}}{U \subset \mathbb{R}^n}$ $\alpha \sim \beta$.

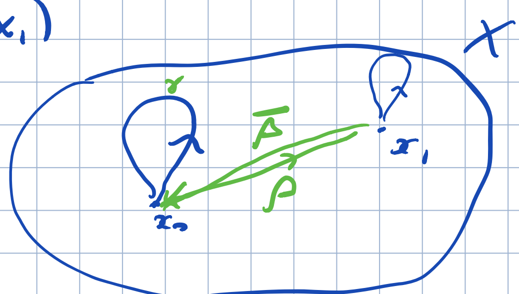
$x = y = x_0$

<u>Ex:</u> if $\underset{\text{convex}}{U \subset \mathbb{R}^n}$ then $\pi_1(U, x_0) = 0$ trivial group, $x_0 \in U$

$\pi_1(X, x_0)$

<u>Lemma</u> Let X a top. space and β a path from x_0 to x_1.
Then $\Phi : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$

$[\gamma] \longmapsto [\bar{\beta} * \gamma * \beta]$



<u>- is an isomorphism</u> of groups.

In particular, X is path connected, the iso class of $\pi_1(X, x_0)$ does not depend on the choice of base point.

<u>Proof</u>: • Φ is <u>well defined</u>:

$\{\gamma_t\}$ - homotopy of loops based at x_0 then $\bar{\beta} * \gamma_t * \beta$

homomorphism — based at x_1

- $\Phi([\gamma_1]\cdot[\gamma_2]) \overset{?}{=} \Phi([\gamma_1])\cdot\Phi([\gamma_2])$

$$\Phi([\gamma_1 * \gamma_2]) = [\bar\beta * \gamma_1 \underset{\substack{\wedge \\ c_{x_0}\sim\beta*\bar\beta}}{*} \gamma_2 * \beta] = [(\bar\beta*\gamma_1*\beta)*(\bar\beta*\gamma_2*\beta)]$$

$$= [\bar\beta*\gamma_1*\beta]\cdot[\bar\beta*\gamma_2*\beta] = \Phi([\gamma_1])\cdot\Phi([\gamma_2]) \quad ✓$$

- iso? $\quad \Phi' : \underset{\text{based at } x_1}{[\gamma']} \longmapsto \underset{\text{based at } x_0}{[\beta*\gamma'*\bar\beta]}$ — inverse of Φ. ✓

$$\Phi'\circ\Phi : [\gamma] \to \Phi'[\bar\beta*\gamma*\beta] = [\underbrace{\beta*\bar\beta}_{\sim c_{x_0}}*\gamma*\underbrace{\beta*\bar\beta}_{c_{x_0}}] = [\gamma]$$

$\Phi'\circ\Phi = id$. ☐

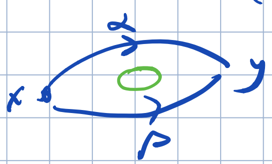
A space X is "simply connected" if it is
- path connected and
- $\pi_1(X, x_0) = 0$

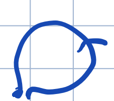
<u>Lemma</u> X is simply connected iff $\forall\, x,y\in X$ $\exists!$ homotopy class of paths from x to y.

<u>Proof</u>: path connectedness $\longleftrightarrow$ existence

suppose $\boxed{\pi_1(X)=0}$, α, β from x to y

then $\alpha \sim (\underbrace{\alpha*\bar\beta}^{\sim c_x})*\beta \sim \beta$.
$\qquad\qquad\qquad\underbrace{\bar\beta*\beta}_{\sim c_y}$



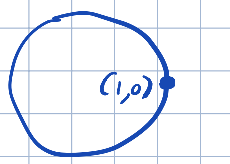
<u>reverse</u>: take $x=y=x_0$  $\sim e_{x_0}$ $\Rightarrow \pi_1(X)=0$ ☐

<u>Fundamental group of the circle</u>

Theorem: $\pi_1(S^1) = \mathbb{Z}$

<u>Explicitly</u>: $\Phi : \mathbb{Z} \to \pi_1(S^1, (1,0))$ is an <u>isomorphism</u>

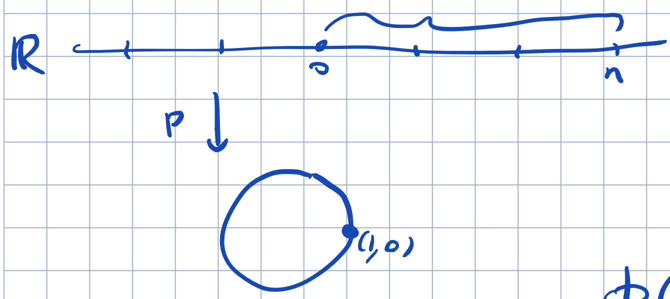
$$n \longmapsto [\omega_n(s) = (\cos 2\pi n s, \sin 2\pi n s)]$$



Rem Φ^{-1} : based loop in S^1 $\longrightarrow \mathbb{Z}$ - winding numbers

<u>Proof</u>: $p: \mathbb{R} \longrightarrow S^1$

$$s \longmapsto (\cos 2\pi s, \ \sin 2\pi s)$$



$$\omega_n = p \circ \tilde{\omega}_n$$

$$I \longrightarrow \mathbb{R}$$

$$s \longmapsto ns$$

$$\Phi(n) = [\omega_n] = [p \circ \tilde{\omega}_n] = \boxed{[p \circ \tilde{\alpha}]}$$

any path from 0 to n in $\mathbb{R}$

• Φ is a homomorphism.

$$\Phi(m+n) = [p(\underbrace{\tilde{\omega}_m * \tau_m(\tilde{\omega}_n)}_{0 \to m+n})] = [\omega_m * \omega_n]$$

$$= \Phi(m) \cdot \Phi(n) \ ✓$$

$\tau_m : s \longmapsto s+m$ translation

• Φ is surjective

<u>Lemma</u> (a)

For each path $\alpha : I \to S^1$ starting at x_0 and each $\tilde{x}_0 \in p^{-1}(x_0)$, there exists a unique <u>lift</u> $\tilde{\alpha} : I \longrightarrow \mathbb{R}$ starting at $\tilde{x}_0$.

i.e. $p \circ \tilde{\alpha} = \alpha$ or

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ I \arrow[ur, dashed, "\tilde{\alpha}"] \arrow[r, "\alpha"'] & S^1 \end{tikzcd}$$

$\alpha : I \to S^1$ loop representing the given class $[\alpha] \in \pi_1(S^1)$ based at $(1,0)$

$\overset{(a)}{\Longrightarrow}$ $\exists! \ \tilde{\alpha}$ - path in $\mathbb{R}$ starting at 0

$\alpha(1) = n \in \mathbb{Z}$

since $p\tilde{\alpha}(1) = \alpha(1) = (1,0)$

$\Rightarrow \Phi(n) = [p \circ \tilde{\alpha}] = [\alpha]$ ✓

- Φ is injective

suppose that $\Phi(m) = \Phi(n)$ $\omega_m \sim \omega_n$

Let α_t be the homotopy $\alpha_0 = \omega_m$ $\alpha_1 = \omega_n$

Lemma (b)

For each homotopy $\alpha_t : I \to S^1$ of paths starting at x_0 and each $\tilde{x}_0 \in p^{-1}(x_0)$, there exists a unique lifted homotopy $\tilde{\alpha}_t : I \to \mathbb{R}$ of paths starting at $\tilde{x}_0$.

i.e. $p \circ \tilde{\alpha}_t = \alpha_t$

$\Rightarrow \exists$ lifted homotopy $\tilde{\alpha}_t$ - a homotopy of paths in $\mathbb{R}$ starting at 0

$\tilde{\alpha}_0 = \tilde{\omega}_m$
$\tilde{\alpha}_1 = \tilde{\omega}_n$ $\leftarrow$ by uniqueness of Lm (b)

$\tilde{\alpha}_t(1)$ is independent of t (since it is a homotopy of paths)

$$\underset{\substack{\shortparallel \\ m}}{\tilde{\alpha}_0(1)} = \underset{\substack{\shortparallel \\ n}}{\tilde{\alpha}_1(1)}$$ ✓

Application: fund. theorem of algebra

every non-constant polynomial with coeff in $\mathbb{C}$ has a root in $\mathbb{C}$.

Proof: We may assume that $p(z) = z^n + a_1 z^{n-1} + \dots + a_n$ $n \neq 0$

assume that p has no roots in $\mathbb{C}$.

$\forall \quad r \geq 0 \qquad \alpha_r(s) = \dfrac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}$ — a loop in $S^1 \subset \mathbb{C}$
unit circle

$0 \leq s \leq 1$

as r varies α_r is a homotopy of loops

$\alpha_0 = c_1 \quad \Rightarrow [\alpha_{r=0}] = 0 \in \pi_1(S^1)$

$\uparrow$ cst loop at 1 $\qquad [\alpha_{r=0}] = [\alpha_r]$

take $r = R \qquad R > \max(1, \sum_i |a_i|)$

for $|z| = R \qquad |z|^n > |a_1 z_1^{n-1} + \dots + a_n|$

$p_t(z) = z^n + t(a_1 z_1^{n-1} + \dots + a_n)$ has no roots on the circle $|z| = R$

$0 \leq t \leq 1$

$$\beta_t(s) = \frac{p_t(re^{2\pi i s})/p_t(r)}{|p_t(re^{2\pi i s})/_t p(r)|}$$

$t \to 0 \qquad \beta_{t=0} = \omega_n \qquad [\omega_n] = n \cdot 1 \neq 0 \in \pi_1(S^1)$

$\uparrow$ generator of $\mathbb{Z}$

$r=0 \rightsquigarrow r=R \qquad t=1 \rightsquigarrow t=0$

$[\alpha_{r=0}] = [\alpha_{r=R}] = [\beta_{t=0}]$

$\| \qquad\qquad\qquad\qquad \|$

$0 \in \pi_1 \qquad\qquad [\omega_n] \neq 0$

$\boxtimes$