**Ex**: Let $X$ be a convex subset of $\mathbb{R}^n$ and $x_0 \in X$. Then (Lemma *) any based loop in $X$ is homotopic to $c_{x_0}$ $\Rightarrow$ group $\pi_1(X, x_0)$ is trivial.

**Lemma** Let $X$ be a top. space and $\beta$ a path from $x_0$ to $x_1$. Then the map

$$\begin{aligned} \Phi: \pi_1(X, x_0) &\longrightarrow \pi_1(X, x_1) \\ [\gamma] &\longmapsto [\bar{\beta} * \gamma * \beta] \end{aligned}$$ is an isomorphism of groups.

In particular, the isomorphism class of $\pi_1(X, x_0)$ of a path connected space does not depend on the choice of base point $x_0 \in X$.

**Proof**: • for $\{\gamma_t\}$ a homotopy of loops based at $x_0$, $\{\bar{\beta} * \gamma_t * \beta\}$ - a homotopy of loops based at $x_1$
$\Rightarrow$ $\Phi$ well-defined

• $\Phi([\gamma_1])\Phi([\gamma_2]) = [\bar{\beta} * \gamma_1 * \beta] \cdot [\bar{\beta} * \gamma_2 * \beta] = [\bar{\beta} * \gamma_1 * \underbrace{\beta * \bar{\beta}}_{\sim c_{x_0}} * \gamma_2 * \beta]$
$= [\bar{\beta} * (\gamma_1 * \gamma_2) * \beta] = \Phi([\gamma_1] \cdot [\gamma_2])$ $\Rightarrow$ $\Phi$ homomorphism

• $\Phi$ is an iso with inverse $\Phi': \underset{\pi_1(X, x_1)}{[\gamma']} \longmapsto [\beta * \gamma' * \bar{\beta}]$ :

$\Phi' \circ \Phi : [\gamma] \longmapsto \Phi'([\bar{\beta} * \gamma * \beta]) = [\underbrace{\beta * \bar{\beta}}_{\sim c_{x_0}} * \gamma * \underbrace{\beta * \bar{\beta}}_{\sim c_{x_0}}] = [\gamma]$ ☑

• A space $X$ is "simply connected" if it is path connected and $\pi_1(X, x_0) = 0$
any point

**Lemma** $X$ is simply connected iff $\forall x, y \in X$ there exists a unique homotopy class of paths $x \to y$.

Proof: existence of a path $\Leftrightarrow X$ path connected

suppose $\pi_1(X)=0$ and $\alpha,\beta$ two paths between $x,y$. Then $\alpha \sim \overbrace{\alpha * \tilde\beta}^{\sim c_y \text{ using } \pi_1=0} * \beta \sim \beta$, where $\underbrace{\tilde\beta * \beta}_{\sim c_x}$

Conversely: if there is a unique homotopy class of paths $x \to x$, then $\pi_1(X,x)=0$. ☐

**Rem** $\Phi^{-1}$: loop $\gamma \mapsto$ winding number $W(\gamma)\in\mathbb{Z}$ in $S^1$

## Fundamental group of the circle.

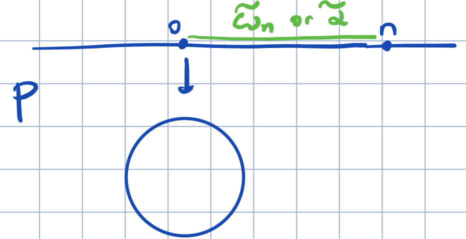
**Theorem**: $\boxed{\pi_1(S^1)=\mathbb{Z}.}$

**Explicitly**: the map $\Phi:\mathbb{Z}\longrightarrow\pi_1(S^1,(1,0))$ is an isomorphism.

$$n \longmapsto [\,\omega_n(s)=(\cos 2\pi n s, \sin 2\pi n s)\,]$$

Proof: • Let $p:\mathbb{R}\twoheadrightarrow S^1$

$s\mapsto(\cos 2\pi s,\sin 2\pi s)$




$\omega_n = p\circ\tilde\omega_n$ $\quad \Phi(n)=[\omega_n]=[p\circ\tilde\omega_n]=[p\circ\tilde\alpha]$ @

$\tilde\omega_n: I\to\mathbb{R}$

$s\longmapsto ns$

any path on $\mathbb{R}$ from 0 to $n$, since $\tilde\omega_n\sim\tilde\alpha$ by linear homotopy.

• $\Phi$ is a homomorphism:

$$\Phi(m+n)=[p(\overbrace{\tilde\omega_m * \underbrace{\tau_m}_{\text{translation}}\tilde\omega_n}^{\text{path } 0\to m+n})]=[\omega_m*\omega_n]=\Phi(m)\cdot\Phi(n) \quad ✓$$

$\tau_m:\mathbb{R}\to\mathbb{R}$

$s\longmapsto s+m$

**Lemma (a)** $\forall$ path $\alpha: I\to S^1$ starting at $x_0\in S^1$ and each $\tilde x_0\in p^{-1}(x_0)$, $\exists!$ lift $\tilde\alpha: I\to\mathbb{R}$ starting at $\tilde x_0$

• $\Phi$ is surjective: let $\alpha: I\to S^1$ loop representing a given class $[\alpha]\in\pi_1(S^1)$

$\alpha(0)=\alpha(1)=(1,0)$

$\underset{Lm(a)}{\Rightarrow}$ $\exists$ lift $\tilde\alpha: I\to\mathbb{R}$ with $\tilde\alpha(0)=0$. Then $\tilde\alpha(1)=n\in\mathbb{Z}$ since $p\,\tilde\alpha(1)=\alpha(1)=(1,0)$

$\Rightarrow \Phi(n)=[p\circ\tilde\alpha]=[\alpha]$ ✓ @

**Lemma (b)** $\forall$ homotopy $\alpha_t: I\to S^1$ of paths starting at $x_0$ and each $\tilde x_0\in p^{-1}(x_0)$ $\exists!$ lifted homotopy $\tilde\alpha_t: I\to\mathbb{R}$ of paths starting at $\tilde x_0$

• $\Phi$ is injective: suppose $\Phi(m)=\Phi(n)$, i.e. $\omega_m\sim\omega_n$

Let $\alpha_t$ be the homotopy: $\alpha_0=\omega_m$, $\alpha_1=\omega_n$

$\underset{Lm(b)}{\Rightarrow}$ it lifts to $\tilde\alpha_t$ - a homotopy of paths in $\mathbb{R}$ starting at 0.

$\tilde\alpha_0=\tilde\omega_m$, $\tilde\alpha_1=\tilde\omega_n$ - by uniqueness in Lm (a)

$\tilde\alpha_t(1)$ is indep. of $t$ (since a homotopy of paths) $\Rightarrow$ $\underset{m}{\underset{\shortparallel}{\tilde\alpha_0(1)}} = \underset{n}{\underset{\shortparallel}{\tilde\alpha_1(1)}}$ ✓ ☐

Lemmata (a), (b) follow from

cover next time

**Lemma (c):** Given a map $F: Y \times I \to S^1$ (top space) and a lifting
$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ Y\times\{0\} \arrow[r, "F|_{Y\times\{0\}}"'] \arrow[ur, dashed, "\tilde F"] & S^1 \end{tikzcd},$$
$\exists!$ lifting
$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ Y\times I \arrow[r, "F"'] \arrow[ur, dashed, "\tilde F"] & S^1 \end{tikzcd}$$
restricting to the given $\tilde F$ on $Y \times \{0\}$

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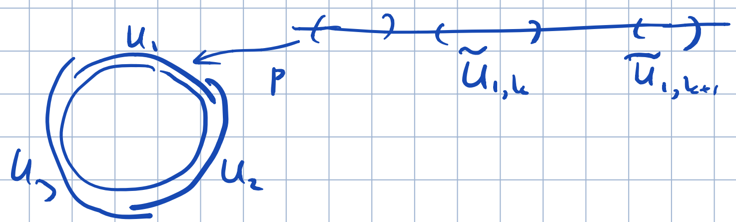
- taking $Y =$ point, we get Lm (a)
- taking $Y = I$, we get Lm (b)

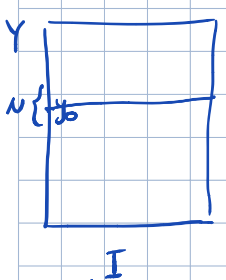
**Proof of Lm (c)** Use that $\exists$ an open cover $\{U_\alpha\}$ of $S^1$ s.t. $\forall \alpha$ $\quad p^{-1}(U_\alpha) \simeq \coprod_j \tilde U_{\alpha, j} \overset{p}{\underset{\approx}{\longrightarrow}} U_\alpha$ (*)

($U_\alpha$'s are "evenly covered" by $p$)

Ex:




- construct the lift locally on $Y$, i.e. a nbhd $N \subset Y$, $N \ni y_0$




$(t, y_0)$ has a nbhd $N_t \times (a_t, b_t) \subset Y \times I$ s.t. $F(N_t \times (a_t, b_t)) \subset U_\alpha$ for some $\alpha$

$\{y_0\}\times I$ cpt $\Rightarrow$ fin. many $(a_t, b_t)$'s cover $\{y_0\}\times I$

$\Rightarrow$ can choose $N \subset Y$, $N \ni y_0$ and a partition $0 = t_0 < t_1 < \dots < t_m = 1$

s.t. $F(N \times [t_i, t_{i+1}]) \subset U_{\alpha_i} =: U_i$

Induction - Assume $\tilde F$ is constructed on $N \times [0, t_i]$. We know that $F(N \times [t_i, t_{i+1}]) \subset U_i$

$\overset{(*)}{\Rightarrow} \exists \tilde U_{i,r} \subset \mathbb{R}$ containing the point $\tilde F(y_0, t_i)$ $\Rightarrow$ $\tilde F(N \times \{t_i\}) \subset \tilde U_{i,r}$
$\tilde U_{i,r} \overset{p}{\underset{\approx}{\to}} U_i$ — replacing $N$ with a smaller nbhd

$\Rightarrow$ define $\tilde F$ on $N \times [t_i, t_{i+1}]$ as $p^{-1} F|_{N\times[t_i, t_{i+1}]}$. ($p^{-1}: \tilde U_{i,r} \leftarrow U_i$)

After finitely many repetitions, we get $\tilde F|_{Y \times I}$ ✓.

Uniqueness of the lift for $Y = pt$ Let $\tilde F, \tilde F'$ be two lifts of $F: I \to S^1$, $\tilde F(0) = \tilde F'(0)$.

as before: $0 = t_0 < t_1 < \dots < t_m = 1$ s.t. $F([t_i, t_{i+1}]) \subset U_i$ some

Induction: assume $\tilde F = \tilde F'$ on $[0, t_i]$.

$[t_i, t_{i+1}]$ connected $\Rightarrow \tilde F([t_i, t_{i+1}])$ connected $\Rightarrow \tilde F([t_i, t_{i+1}]) \subset \tilde U_{i,r}$ for some $r$ ($\tilde U_{i,r} \overset{p}{\underset{\approx}{\to}} U_i$)

similarly $\tilde F'([t_i, t_{i+1}]) \in \tilde U_{i,r'}$

$\tilde F(t_i) = \tilde F'(t_i) \Rightarrow r = r' \underset{\substack{\text{since } p\tilde F = p\tilde F' \\ \text{and } p \text{ injective on } \tilde U_{i,r}}}{\Rightarrow} \tilde F = \tilde F'$ on $[t_i, t_{i+1}]$ ✓

- $\tilde F$ constructed on $N \times I$ are unique when restricted to $\{y\} \times I$ $\Rightarrow$ must agree when $N \times I$ and $N' \times I$ overlap $\Rightarrow$ we have a well-defined lift $\tilde F$ on $Y \times I$. $\boxed{\checkmark}$

//

application:

**Fundamental theorem of algebra**

Every nonconstant polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.

Proof: We may assume $p(z) = z^n + a_1 z^{n-1} + \dots + a_n$. Assume $p$ has no roots in $\mathbb{C}$.

$$\forall r \geq 0, \quad \alpha_r(s) \overset{(\ast\ast)}{=} \frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}$$

defines a loop in the unit circle $S^1 \subset \mathbb{C}$ based at $1$.

as $r$ varies, $f_r$ is a homotopy of loops.

- $\alpha_0 = \underset{\text{const loop at } 1}{C_1} \Rightarrow [\alpha_r] = 0 \in \pi_1(S^1)$ $\forall r$

- take $r$ large $\left( r > \max(1, \sum_i |a_i|) \right)$.

  for $|z| = r$, $\quad |z^n| = r^n = r \cdot r^{n-1} > \left(\sum_{i=1}^n |a_i|\right) |z|^{n-1} \geq |a_1 z^{n-1} + \dots + a_n|$

  $\Rightarrow$ polynomial $p_t(z) = z^n + t(a_1 z^{n-1} + \dots + a_n)$ has no roots on the circle $|z| = r$ for $0 \leq t \leq 1$.

Replacing $p$ with $p_t$ in $(\ast\ast)$ and letting $t \in [0,1]$,

we obtain a homotopy between $\underset{t=1}{\alpha_r}$ and the loop $\underset{t=0}{\omega_n(s) = e^{2\pi i n s}}$

$\alpha_r,\ 0 \leq r \leq R$ $\quad \beta_t,\ 1 \geq t \geq 0$

$\alpha_0 \sim \alpha_{r=R} \sim \omega_n$

$[\omega_n] = n \cdot \underset{\substack{\uparrow \\ \text{generator of } \mathbb{Z} \cong \pi_1(S^1)}}{1} \in \pi_1(S^1) \Rightarrow \underset{(p = \text{const})}{n = 0}$ $\boxed{\checkmark}$

$\quad \| $

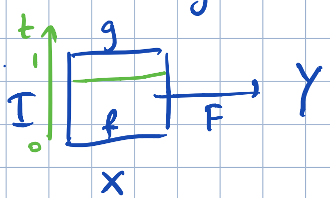
$\quad 0$

**Induced map on $\pi_1$** 

- Let $f: X \to Y$ cont. map, $\gamma, \delta$ - paths in $X$. Then:
  - if $\gamma$ and $\delta$ have same endpoints and $\gamma \sim \delta$, then $f \circ \gamma \sim f \circ \delta$ — Compatibility with homotopies
  - if $\gamma(1) = \delta(0)$, then $f \circ (\gamma * \delta) = (f \circ \gamma) * (f \circ \delta)$ — compat. with $*$.

<u>def</u> Let $f: X \to Y$, $x_0 \mapsto y_0$ cont. map. The map
$$f_*: \pi_1(X, x_0) \to \pi_1(Y, y_0), \quad [\gamma] \mapsto [f \circ \gamma]$$
is called the map of fund. groups induced by $f$.

- two cont. maps $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ are called "homotopic" ($f \sim g$) if $\exists$ a cont. map $H: X \times I \to Y$ s.t. $H(x,0) = f(x)$, $H(x,1) = g(x)$



= "interpolating family of maps" $f_t: X \to Y$

- two top spaces $X, Y$ are called <u>homotopy equivalent</u> (or "homotopic") if $\exists f, g$, $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ s.t. $g \circ f \sim id_X$, $g \circ f \sim id_Y$

Ex: $\mathbb{C} \setminus \{0\} \underset{g}{\overset{f}{\rightleftarrows}} S^1$ ($g$ ← inclusion) $\quad f: z \mapsto \frac{z}{|z|}$

we have $f \circ g = id_{S^1}$, $\underset{z \mapsto \frac{z}{|z|}}{g \circ f} \sim id_{\mathbb{C} \setminus \{0\}}$ with $H(t, z) = |z|^t \frac{z}{|z|}$ — homotopy fixing basepoints

<u>Lemma</u>: if $f: (X, x_0) \to (Y, y_0)$ homotopy equivalence of <u>pointed</u> top spaces ← base points!
then $f_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an <u>isomorphism</u>.

Ex: $\pi_1(\mathbb{C} \setminus \{0\}) \simeq \pi_1(S^1) \simeq \mathbb{Z}$