chain rule reminder:

$$a \in U \xrightarrow{F} \overset{F(a)}{V} \xrightarrow{G} W$$
$$U \subset \mathbb{R}^m,\quad V \subset \mathbb{R}^n,\quad W \subset \mathbb{R}^k$$
$$\mathbb{R}^m \xrightarrow[DF_a]{} \mathbb{R}^n \xrightarrow[DG_{F(a)}]{} \mathbb{R}^k$$

chain rule: $D(G\circ F)_a = DG_{F(a)} \circ DF_a$

or: $\dfrac{\partial z_\alpha}{\partial x_i} = \sum \dfrac{\partial z_\alpha}{\partial y_a}\dfrac{\partial y_a}{\partial x_i}$

<source of examples of manifolds>

Thm

Let $F: U \to \mathbb{R}^m$, $U \subset \mathbb{R}^{n+m}$ open, $F = (F_1, \dots, F_m)$, be a C^∞ function, fix $c \in \mathbb{R}^m$. Assume that $\forall a \in F^{-1}(c)$, the derivative $DF_a : \mathbb{R}^{n+m} \to \mathbb{R}^m$ is surjective. Then $M = F^{-1}(c) \subset \mathbb{R}^{n+m}$ has the structure of a smooth n-manifold.

$$DF_a = \left(\frac{\partial F_i}{\partial x_j}\right)\Big|_{x=a}$$

recall: $F(a+h) = F(a) + DF_a(h) + R(a,h)$ with $R(a,h)/\|h\| \xrightarrow[h\to 0]{} 0$

Proof: DF_a is surjective $\Leftrightarrow$ matrix $\left(\dfrac{\partial F_i}{\partial x_j}\right)\Big|_{\substack{x=a\\ 1\le i\le m\\ 1\le j\le n+m}}$ has rank $= m$

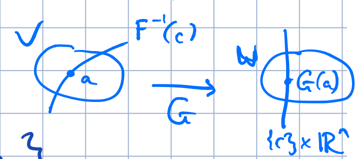
$\Rightarrow$ by reordering the coordinates $x_1, \dots, x_{n+m}$, we may assume that the square matrix $\left(\dfrac{\partial F_i}{\partial x_j}\right)_{\substack{1\le i\le m\\ 1\le j\le m}}$ is invertible. Define $G: U \to \mathbb{R}^{n+m}$

$$(x_1 \dots x_{n+m}) \mapsto (F_1, \dots, F_m, x_{m+1}, \dots, x_{n+m})$$

$$DG_a = \begin{pmatrix} \text{invertible} & * \\ 0 & I \end{pmatrix}\begin{matrix}\}m\\ \}n\end{matrix}$$

$\Rightarrow DG_a$ is invertible

By inverse function theorem, $\exists\, V \subset \mathbb{R}^{n+m}$ open, $V \ni a$, $W \subset \mathbb{R}^{n+m}$ open, $W \ni G(a)$, s.t. $G: V \to W$ is invertible, with smooth inverse



G maps $V \cap F^{-1}(c)$ to $\underbrace{(\{c\}\times\mathbb{R}^n)}_{\text{copy of } \mathbb{R}^n \text{ given by } \{x\in\mathbb{R}^{n+m};\ x_1 = c_1, \dots, x_m = c_m\}} \cap W$

$\Rightarrow \underbrace{p\circ G}_{\varphi}: V \cap F^{-1}(c) \to \mathbb{R}^n$ is a coord. chart on $M = F^{-1}(c)$.

$V_\alpha \xrightarrow{G_\alpha} W_\alpha$

$V_\beta \xrightarrow{G_\beta} W_\beta$

Given two such charts $\varphi_\alpha, \varphi_\beta$, $G_\alpha G_\beta^{-1}$ - C^∞-map between open sets in $\mathbb{R}^{n+m}$

$\Rightarrow \varphi_\alpha \varphi_\beta^{-1} = p\, G_\alpha G_\beta^{-1}\, \underset{\text{inclusion}}{i}$ is C^∞.

$\Rightarrow$ we have an atlas.

$$\text{inclusion } \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+m} = \mathbb{R}^m \times \mathbb{R}^n,\quad x \longmapsto (c, x)$$

- $\mathbb{R}^{n+m}$ is Hausdorff, 2nd countable $\Rightarrow M = F^{-1}(c) \subset \mathbb{R}^{n+m}$ is, too $\square$

Ex: ① $F: \mathbb{R}^{n+1} \to \mathbb{R}$, $x \mapsto \|x\|^2$, $F^{-1}(1) = S^n$
(1 ← reg. value of F)

② $F: \mathrm{Mat}_{n\times n} \to \mathrm{Sym\,Mat}_{n\times n}$, $A \mapsto A^TA$, $F^{-1}(I) = O(n)$
(I ← reg. value of F)

<back to>
Smooth functions

$f: M \to \mathbb{R}$ C^∞ — can add and multiply by constants $\Rightarrow$ they form a vector space $C^\infty(M)$

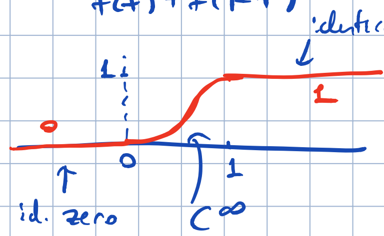
— can multiply themselves $\Rightarrow$ form a commutative ring.

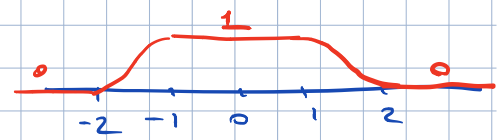
* There are many smooth functions on a smooth mfd M.

· **Bump function**

one-variable:

$$f(t) = \begin{cases} e^{-1/t}, & t>0 \\ 0, & t \le 0 \end{cases} \qquad g(t) = \frac{f(t)}{f(t)+f(1-t)} \qquad h(t) = g(2+t)\,g(2-t) \begin{cases} =0 & \text{if } |t|\geq 2 \\ =1 & \text{if } |t| \le 1 \end{cases}$$





• n-dim. version

~~$k(x_1,\dots,x_n) = h(x_1)\cdots h(x_n)$ (*) $k\left(\frac{x}{r}\right) = \begin{cases} 1 & \text{in a ball of radius } r \\ 0 & \text{outside a ball of radius } 2\sqrt{n}\, r \end{cases}$~~

BETTER: $k(x_1,\dots,x_n) = h(\|x\|) = h\left(\sqrt{x_1^2+\dots+x_n^2}\right) \Rightarrow k\left(\frac{x}{r}\right) = \begin{cases} 1 & \text{if } \|x\| \le r \\ 0 & \text{if } \|x\| \geq 2r \end{cases}$

• let (U, φ_U) a coord. chart on M

choose a function k of the type (*) s.t. $\operatorname{supp} k = \overline{\{x : k(x) \neq 0\}}$ lies in $\varphi_U(U)$

and set
$$f: M \to \mathbb{R}, \quad x \mapsto \begin{cases} k \circ \varphi_U(x), & x \in U \\ 0, & x \in M \setminus U \end{cases}$$

$f \in C^\infty(M)$: f is C^∞ in U. $\operatorname{supp} k \subset \mathbb{R}^n$ closed, bounded $\Rightarrow$ compact

$\Rightarrow \operatorname{supp} f \subset M$ compact

$\Rightarrow f = 0$ in $\underbrace{M}_{\text{Hausdorff}} \setminus \underbrace{\operatorname{supp} f}_{\text{cpt} \Rightarrow \text{closed}}$ = open in M $\Rightarrow$ $f=0$ in a nbhd of any pt $x \in M \setminus \operatorname{supp} f$

$\Rightarrow f$ is smooth in $M \setminus \operatorname{supp} f$

$\Rightarrow f \in C^\infty(M)$ □

Derivative of a function

$f \in C^\infty(M)$ when does a derivative at a vanish? (e.g. f has a maximum at a)

$g = f \circ \underbrace{\varphi_\alpha^{-1}}_{\text{coord. chart}} \in C^\infty(\varphi_\alpha(U_\alpha))$. Suppose $Dg|_{\varphi_\alpha(a)} = 0$.   Let $h = f \circ \varphi_\beta^{-1}$

$Dg = (\partial_i g)_{i=1\dots n}$ $g = h \circ \varphi_\beta \varphi_\alpha^{-1}$

$\Rightarrow g(x_1 .. x_n) = h(y_1(x), .., y_n(x))$

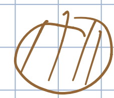
$\Rightarrow \frac{\partial g}{\partial x_i} = \sum_j \frac{\partial h}{\partial y_j}(y(x)) \underbrace{\frac{\partial y_j}{\partial x_i}(x)}_{\substack{\text{invertible matrix,} \\ \text{since } y(x) \text{ is invertible}}}$ $\Rightarrow Dg|_{x(a)} = 0$ iff $Dh|_{y(a)} = 0$

chain rule

$\Rightarrow$ vanishing of the derivative at a is independent of the coord. chart.

Let $Z_a = \{ f \in C^\infty(M) \mid f$ has vanishing derivative at $a \} \subset C^\infty(M)$
vect. subspace

def The __cotangent space__ T_a^* at $a \in M$ is the quotient space $T_a^* = C^\infty(M) / Z_a$. The derivative of $f \in C^\infty(M)$ at $a \in M$ is its image in this space and is denoted $(df)_a$.



- if $f \in C^\infty(M)$, $(df)_a = d(\mu \cdot f)_a$ $\Rightarrow$ can make sense of $(df)_a$ for a locally-defined smooth f (in a nbhd of a), such as $f = x_1, .., x_n$ - loc. coord. functions

since if v vanishes in nbhd of a, then $v \in Z_a \Rightarrow (dv)_a = 0$. set $v = f - \mu f$

bump function $\equiv 1$ in the nbhd of a

__Proposition__: Let M be an n-mfd. Then

- The cotangent space T_a^* at $a \in M$ is an n-dimensional vector space.
- If (U, φ) is a coord. chart around a with coords $x_1, .., x_n$, then the elements $(dx_1)_a, .., (dx_n)_a$ form a basis for T_a^*.
- If $f \in C^\infty(M)$ and in the coord. chart, $f \circ \varphi^{-1} = \psi(x_1, .., x_n)$ then

$$(df)_a = \sum_i \frac{\partial \psi}{\partial x_i}(\varphi(a)) (dx_i)_a \qquad (*)$$

__Proof__ $f - \sum_i \frac{\partial \psi}{\partial x_i}(\varphi(a)) x_i$ - locally-defined smooth function whose derivative vanishes at a.

$\Rightarrow (df)_a = \sum \frac{\partial \psi}{\partial x_i}(\varphi(a)) (dx_i)_a$

and $(dx_i)_a$ span T_a^*. If $\sum_i \lambda_i (dx_i)_a = 0$ then $\sum_i \lambda_i x_i$ has vanishing derivative at $a \Rightarrow \lambda_1 = .. = \lambda_n = 0$.

__Rem__ We will denote $\psi = f$.
$\uparrow$ coord. representation of f

So that $(*)$ becomes: $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$.

With a change of coord. $(x_1, \dots, x_n) \longmapsto (y_1(x), \dots, y_n(x))$, we get

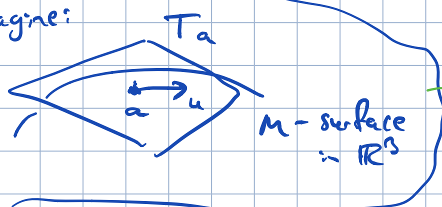
$$df = \sum_j \frac{\partial f}{\partial y_j} dy_j = \sum_{i,j} \frac{\partial f}{\partial y_j} \frac{\partial y_j}{\partial x_i} dx_i$$

* def The <u>tangent space</u> T_a at $a \in M$ is the dual space to the cotangent space T_a^*.

- if $x_1, \dots, x_n$ - loc. coord. system at a and $(dx_1)_a, \dots, (dx_n)_a$ - the corresp. basis of T_a^*; the dual basis of T_a is denoted $\left(\frac{\partial}{\partial x_1}\right)_a, \dots, \left(\frac{\partial}{\partial x_n}\right)_a$

two approaches to intrinsic definition of T_a: (i) equivalence classes of curves $f: \mathbb{R} \to M$

imagine:



(ii) tangent vector $\vec{u} \rightsquigarrow$ directional derivative $f \longmapsto \vec{u} \cdot \nabla f(a)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{Df_a(u)}$

<algebraic definition>

def A tangent vector at $a \in M$ is a linear map $X_a : C^\infty(M) \to \mathbb{R}$ s.t.

$$\boxed{X_a(fg) = f(a) X_a g + g(a) X_a f}.$$

(formal Leibnitz rule)

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if $\underbrace{\xi \in T_a}_{\text{dual space to } C^\infty(M)/Z_a}$, $\quad f \longmapsto \xi((df)_a)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad C^\infty(M) \longrightarrow \mathbb{R}$

Moreover, from (*): $d(fg)_a = f(a)(dg)_a + g(a)(df)_a$

(#) Thus, $X_a : f \longmapsto \xi((df)_a)$ is a tangent vector.

- In fact, all tangent vectors are of this form!

injective! map $T_a \to T_a^{Alg}$ (cannot have a $\xi \neq 0$ s.t. $\xi((df)_a) = 0 \ \forall f$)

$T_a \longrightarrow T_a^{Alg}$
$(\xi : T_a^* \to \mathbb{R}) \longleftarrow X_a$
$[df]_a \mapsto X_a f$
or just: $(X_a : C^\infty(M)/Z_a \to \mathbb{R}) \longleftarrow X_a$

Lemma Let X_a be a tangent vector at a and $f \in Z_a$. Then $X_a f = 0$.

Proof. Use a loc. coord. sys near a:

$$f(x) - f(a) = \int_0^1 \frac{\partial}{\partial t} f(a + t(x-a))\, dt = \sum_i (x_i - a_i) \underbrace{\int_0^1 \frac{\partial f}{\partial x_i}(a + t(x-a))\, dt}_{g_i(x)}$$

fund thm of calc

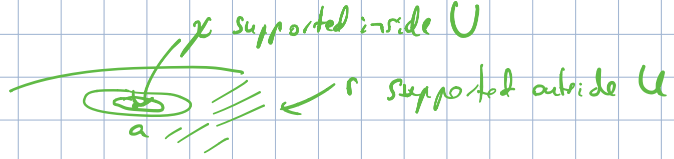
if $(df)_a = 0$, then $g_i|_{x=a} = 0$ and $h_i(x) = x_i - a_i$ also vanishes at $x = a$.

$f = f(a) + \sum_i g_i h_i$ - locally, near a. with g_i, h_i vanishing at $x=a$.

$\Big\{ \to f = f(a) + \sum_i \tilde{g}_i \tilde{h}_i + \underbrace{r}$ - globally

$\psi g_i \quad \psi h_i$ vanishes in nbhd of a

global extension by a bump fun. ψ



bump near a

• $\underset{0}{X_a(r \cdot \cancel{\chi})} = \underbrace{\cancel{\chi(a)}}_{1} X_a(r) + \underbrace{r(a)}_{0} X_a(\cancel{\chi}) = X_a(r) \Rightarrow X_a(r) = 0$

• $X_a(1 \cdot 1) \underset{\uparrow}{=} 1 \cdot X_a(1) + 1\, X_a(1) \Rightarrow X_a(1) = 0 \Rightarrow X_a\left(\text{any const function}\right) = 0$

Leibnitz

$\Rightarrow \underline{X_a f = \sum_i \tilde{g}_i(a) X_a(\tilde{h}_i) + \tilde{h}_i(a) X_a(\tilde{g}_i)} = 0$ $\quad\boxtimes$

• if $V \subset W$ vector spaces, $\mathrm{Ann}(V) \subset W^*$ annihilator,
then $\mathrm{Ann}(V) \cong (W/V)^*$

set $W = C^\infty(M)$, $V = Z_a$ then $\left(C^\infty(M)/Z_a\right)^* = T_a$

$\mathrm{Ann}(Z_a) \subset \left(C^\infty(M)\right)^*$, with $\left(C^\infty(M)/Z_a\right)^* \cong \mathrm{Ann}(Z_a) \supseteq \{\text{tangent vectors}\} =: T_a^{alg}$ Lemma

$\Rightarrow T_a^{alg} \subseteq T_a$

- together with (#), it gives $T_a \cong T_a^{alg}$.

Thus, vectors in T_a are the tangent vectors

Locally; in coordinates: $X_a = \sum_{i=1}^{n} c_i \left(\frac{\partial}{\partial x_i}\right)_a$

then $X_a f = \sum_{i=1}^{n} c_i \frac{\partial f}{\partial x_i}(a)$