Ex: $\underset{\{\dots,a^{-1},1,a,a^2,\dots\}}{\mathbb{Z}} * \underset{\{\dots,b^{-1},1,b,b^2,\dots\}}{\mathbb{Z}} = \{\text{words in } a,a^{-1},b,b^{-1}\} / \sim \;= \left\{ x_1^{n_1}\cdots x_k^{n_k} \;\middle|\; \begin{array}{l} n_k \neq 0 \\ \text{either } x_1=a, x_2=b, x_3=a, x_4=b \text{ etc} \\ \text{or } x_1=b, x_2=a \text{ etc.} \end{array} \right\} \cup \{1\}$

$= \langle a, b \rangle$ – "free group with two generators a, b"

- $\mathbb{Z}_2 * \mathbb{Z}_2 = ?$ e.g. is it a finite group?

  $\mathbb{Z}_2 = \{1,a\} = \langle a \rangle / a^2 = \langle a \mid a^2 = 1 \rangle$, $\mathbb{Z}_2 = \langle b \mid b^2 = 1 \rangle$

  quotient by normal subgroup generated by a^2; generator; relation

  $\mathbb{Z}_2 * \mathbb{Z}_2 = \langle \underbrace{a, b}_{\text{generators}} \mid \underbrace{a^2 = 1,\ b^2 = 1}_{\text{relations}} \rangle$

- Universal property: if $G_1 \xrightarrow{f_1} H$, $G_2 \xrightarrow{f_2} H$ are group homomorphisms (for some H),

  then $\exists!$ group hom $f: G_1 * G_2 \longrightarrow H$ st

  diagram
$$\begin{tikzcd} G_1 \arrow[d, "i_1"'] \arrow[dr, "f_1"] & \\ G_1 * G_2 \arrow[r, dashed, "f"] & H \\ G_2 \arrow[u, "i_2"] \arrow[ur, "f_2"'] & \end{tikzcd}$$
  is commutative: $f_1 = f \circ i_1$, $f_2 = f \circ i_2$

  <proof → homework>

Amalgamated free product.

def Let G_1, G_2, A be groups, $j_1 : A \to G_1$, $j_2 : A \to G_2$ homomorphisms.

Then: $G_1 \underset{A}{*} G_2 := G_1 * G_2 \Big/ \begin{array}{l}\text{normal subgroup} \\ \text{generated by elements} \\ j_1(\alpha)\, j_2(\alpha)^{-1} \ \forall \alpha \in A\end{array} = \dfrac{\text{words of elements of } G_1, G_2}{\sim \text{ as before, plus } \dots j_1(\alpha)\dots \sim \dots j_2(\alpha)\dots \ \forall \alpha \in A}$

"amalgamated free product"

Ex:

- $G_1 \underset{1}{*} G_2 = G_1 * G_2$ – usual free product

  (1: triv. group)

- $\langle a_1, \dots, a_k \mid \text{relations}_1 \rangle \underset{A}{*} \langle b_1, \dots, b_\ell \mid \text{relations}_2 \rangle$

  $= \langle a_1, \dots, a_k, b_1, \dots, b_\ell \mid \text{relations}_1, \text{relations}_2, \{ j_1(\alpha) = j_2(\alpha) \}_{\alpha \in A} \rangle$

  suffices to put only the generators of A here.

- $\underset{\substack{\uparrow \\ \langle a\rangle}}{\mathbb{Z}} * \underset{\substack{\mathbb{Z} \\ \uparrow \\ \langle b\rangle}}{1} = ?$ if $j_1 : b \mapsto a^k$, $j_2 : b \mapsto 1$

$= \langle a \mid a^k = 1 \rangle = \{1, a, a^2, \dots, a^{k-1}\} \simeq \mathbb{Z}/_k\mathbb{Z}$ the group of residues mod. k.

- notice that it does depend on the maps j_1! (via k)

- $\underset{\substack{\uparrow \\ \langle a,b\rangle}}{(\mathbb{Z} * \mathbb{Z})} * \underset{\substack{\mathbb{Z} \\ \uparrow \\ \langle c\rangle}}{1}$ $\quad c \overset{j_1}{\mapsto} aba^{-1}b^{-1}$, $c \mapsto 1$

$= \langle a, b \mid aba^{-1}b^{-1} = 1 \rangle = \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$

or: $ab = ba$

Using Seifert-van Kampen to compute $\pi_1(X)$ - examples

Recall SvK: $\boxed{\pi_1(X) = \pi_1(X_1) \underset{\pi_1(Y)}{*} \pi_1(X_2)}$ if $X = X_1 \cup X_2$ (open)

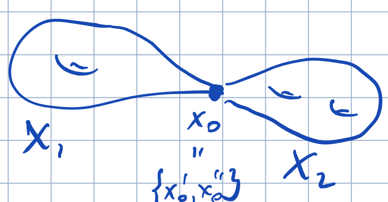
(#) w.r.t. $(j_a)_*$

$Y = X_1 \cap X_2$ - path connected, contains x_0

$j_a : Y \hookrightarrow X_a$

Ex: (1) $X = X_1 \vee X_2$ - "wedge sum of (pointed) spaces $\underset{x_0'}{\overset{\psi}{X_1}}$, $\underset{x_0''}{\overset{\psi}{X_2}}$"

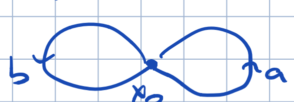
$\overset{\shortparallel}{X_1 \amalg X_2 /_{x_0' \sim x_0''}}$



By (#): $\pi_1(X, x_0) = \pi_1(X_1, x_0') * \pi_1(X_2, x_0'')$

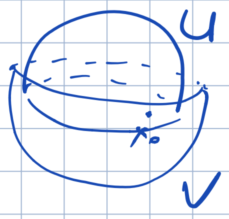
- free product

(1') E.g. $\pi_1(S^1 \vee S^1, x_0) = \mathbb{Z} * \mathbb{Z} = \langle a, b \rangle$



(2) $\pi_1(S^2) = \pi_1(\overset{\approx \text{ open disk}}{U}) \underset{\pi_1(U \cap V)}{*} \pi_1(V) = 1 \underset{\mathbb{Z}}{*} 1 = 1$ - trivial!

$U \cap V \approx S^1 \times (-1, 1)$



$U: 0 \le \theta < \frac{\pi}{2} + \varepsilon$, $\quad V: \frac{\pi}{2} - \varepsilon < \theta \le \pi$

azimuthal angle

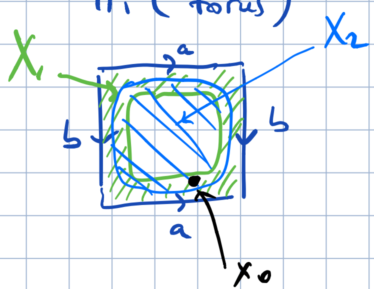
$W = U \cap V : \frac{\pi}{2} - \varepsilon < \theta < \frac{\pi}{2} + \varepsilon$

Similarly: $\underset{n \ge 2}{\pi_1(S^n)} = \pi_1(U) \underset{\pi_1(U \cap V)}{*} \pi_1(V) = 1 * 1 = 1$

extended upper hemisphere $\approx$ open disk

1 or $\mathbb{Z}$: $n \ge 3$, $n = 2$

(9) $\pi_1(\text{torus})$ – via van Kampen (already know it as a π_1 of a product)



$T = X_1 \cup X_2$, $\quad Y = X_1 \cap X_2 \approx S^1 \times (-1,1) \sim S^1$

X_1: nbhd of the based loops a, b ~ wedge of two circles

X_2: open disk ~ pt

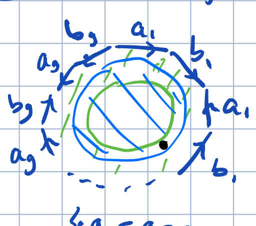
So: $\pi_1(T) = \pi_1(X_1) \underset{\pi_1(Y)}{*} \pi_1(X_2) = (\underset{\langle a,b \rangle}{\mathbb{Z} * \mathbb{Z}}) \underset{\underset{\langle c \rangle}{\mathbb{Z}}}{*} 1 \Leftrightarrow$

with $c \xrightarrow{(j_1)_*} aba^{-1}b^{-1} \in \pi_1(X_1)$

$c \xrightarrow{(j_2)_*} 1 \in \pi_1(X_2)$

$\Leftrightarrow \langle a, b \mid aba^{-1}b^{-1} = 1 \rangle = \mathbb{Z}^2$.

(3') Similarly: $\pi_1(\Sigma_g) = \underset{\langle a_1, b_1, \dots, a_g, b_g \rangle}{\mathbb{Z}^{*2g}} \underset{\underset{\langle c \rangle}{\mathbb{Z}}}{*} 1 = \boxed{\langle a_1, b_1, \dots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1 \rangle}$



4g-gon

with $c \xrightarrow{j_1} a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$

$c \xrightarrow{j_2} 1$

Products, coproducts, pushouts

Products

recall: for X_1, X_2 top spaces, $\begin{tikzcd} & X_1 \times X_2 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X_1 & & X_2 \end{tikzcd}$ projections.

$f: Y \to X_1 \times X_2$ is cont. iff component maps $f_1 = p_1 \circ f : Y \to X_1$ and $f_2 = p_2 \circ f : Y \to X_2$ are cont.

Or: $f: Y \to X_1 \times X_2$ is uniquely determined by a pair $f_1: Y \to X_1$, $f_2: Y \to X_2$ and $f_i = p_i \circ f$

Diagrammatically:

$$\begin{tikzcd} & X_1 \\ Y \arrow[ur, bend left, "f_1"] \arrow[r, dashed, "\exists! f"] \arrow[dr, bend right, "f_2"'] & X_1 \times X_2 \arrow[u, "p_1"'] \arrow[d, "p_2"] \\ & X_2 \end{tikzcd}$$

- given the commut. diagram given by solid arrows, $\exists!$ map f - dashed arrow - making the whole diag. commutative.

(Commutativity in top/bottom triangles $\Leftrightarrow$ f_1, f_2 are components of f)

In any category:

def Let X_1, X_2 be objects in a category C. $X \in Ob(C)$ is the "categorical product" (denoted $X_1 \times X_2$) if there are morphisms $p_1 : X \to X_1$, $p_2 : X \to X_2$ s.t. the diagram $X_1 \xleftarrow{p_1} X \xrightarrow{p_2} X_2$ has the property: $\forall\, Y \in Ob(C)$ and $f_i : Y \to X_i$, $i = 1,2$, $\exists!\, f: Y \to X$ making the diagram

$$\begin{tikzcd} & X_1 \\ Y \arrow[ur, "f_1"] \arrow[r, dashed, "\exists! f"'] \arrow[dr, bend right, "f_2"'] & X \arrow[u, "p_1"'] \arrow[d, "p_2"] \\ & X_2 \end{tikzcd} \qquad (*)$$

Rem 1. In fact, "the" categorical product is defined up to (a unique) isomorphism:

$$\begin{tikzcd} & X_1 & \\ X \arrow[ur, "p_1"] \arrow[dr, "p_2"'] \arrow[rr, shift right, dashed, "g"'] & & X' \arrow[ul, "p_1'"'] \arrow[dl, "p_2'"] \arrow[ll, shift right, dashed, "f"'] \\ & X_2 & \end{tikzcd}$$

$f \circ g = id_X$ by uniqueness in $(*)$ for $Y = X$.

- for $C = Set, Vect, Grp, Top, Top_*$, the categorical product = Cartesian product with usual projection maps $p_i : X_1 \times X_2 \to X_i$ $i = 1,2$

Coproducts.

motivating example: disjoint union of sets

- for X_1, X_2 sets, the disjoint union is the set

$$X_1 \amalg X_2 := \{(x,1) \mid x \in X_1\} \cup \{(x,2) \mid x \in X_2\} \subset (X_1 \cup X_2) \times \{1,2\}$$

$$X_1 \xrightarrow{i_1} X_1 \amalg X_2 \xleftarrow{i_2} X_2 \qquad (*)$$

$$x \longmapsto (x,1) \qquad (x,2) \longmapsfrom x$$

- any map $f: X_1 \amalg X_2 \to Y$ is completely determined by restrictions to X_1, X_2, i.e. by $f_1 = f \circ i_1$, $f_2 = f \circ i_2$.

$$\begin{tikzcd} X_1 \arrow[d, "i_1"'] \arrow[dr, "f_1"] & \\ X_1 \amalg X_2 \arrow[r, dashed, "\exists! f"'] & Y \\ X_2 \arrow[u, "i_2"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

def Let X_1, X_2 be objects in a category $\mathcal{C}$. $X \in Ob(\mathcal{C})$ is called a coproduct of X_1, X_2 (notation: $X_1 \amalg X_2$) if there are morphisms $X_1 \xrightarrow{i_1} X \xleftarrow{i_2} X_2$ s.t. this pair of maps satisfies the univ. property expressed by the comm diag

$$\begin{tikzcd} X_1 \arrow[d, "i_1"'] \arrow[dr, "f_1"] & \\ X \arrow[r, dashed, "\exists! f"'] & Y \\ X_2 \arrow[u, "i_2"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

coproducts may fail to exist.
Ex: $C = \{$sets with 3-elements + maps between them$\}$

Ex: (Coproducts in some categories)

$\mathcal{C}$	coproduct	
Set	$X_1 \amalg X_2$	disj. union
Vect	$X_1 \oplus X_2$	direct sum
Grp	$X_1 * X_2$	free product
Top	$X_1 \amalg X_2$	disj. union
Top_*	$X_1 \vee X_2$	wedge sum (or wedge product)

$$X_1 \vee X_2 = X_1 \amalg X_2 \big/ \{i_1(x_1), i_2(x_2)\}$$

Pushouts

motivating example - in Top

let $X_1, X_2 \underset{\text{open}}{\subset} X$ ← top. sp.

then we have the comm. square

$$\begin{tikzcd} X_1 \cap X_2 \arrow[r, "i_1"] \arrow[d, "i_2"'] & X_1 \arrow[d, "k_1"] \\ X_2 \arrow[r, "k_2"'] & X \end{tikzcd}$$

let $f_1 : X_1 \to Y$, $f_2 : X_2 \to Y$ cont. maps which agree on $X_1 \cap X_2$

Then $\exists!$ well-defined cont. map $f: X \to Y$ s.t. $f|_{X_i} = f_i$ (f is "glued" out of maps f_1, f_2)

def In a cat. $\mathcal{C}$, a comm. diagram of objects & morphisms

$$\begin{tikzcd} A \arrow[r, "j_1"] \arrow[d, "j_2"'] & X_1 \arrow[d, "k_1"] \\ X_2 \arrow[r, "k_2"'] & X \end{tikzcd}$$

is a "pushout diagram" if it satisfies the univ. property expressed by

$$\begin{tikzcd} A \arrow[r, "j_1"] \arrow[d, "j_2"'] & X_1 \arrow[d, "k_1"] \arrow[ddr, bend left, "f_1"] & \\ X_2 \arrow[r, "k_2"'] \arrow[drr, bend right, "f_2"'] & X \arrow[dr, dashed, "\exists! f"] & \\ & & Y \end{tikzcd}$$

The object X is called the pushout of the diagram

$$\begin{tikzcd} A \arrow[r, "j_1"] \arrow[d, "j_2"'] & X_1 \\ X_2 & \end{tikzcd}$$

Ex:

Category	pushout
Set	$X_1 \underset{A}{\cup} X_2$
Top	$X_1 \underset{A}{\cup} X_2$
Top$_*$	$X_1 \underset{A}{\cup} X_2$
Grp	$X_1 \underset{A}{*} X_2$
Vect	$X_1 \oplus X_2 / (j_1 - j_2)(A)$

Set: $:= X_1 \amalg X_2 / i_1(j_1(a)) \sim i_2(j_2(a))$ — from (*)

Grp: amalgamated free product