FINAL EXAM, DUE 11/18/2020 AT 10AM

Problem 1. Prove that the a topology on the finite set $\{1, 2, \ldots, n\}$ is Hausdorff if and only if it is the discrete topology.

Problem 2.

(a) Let M be an oriented n-manifold (without boundary), $\alpha \in \Omega_c^n(M)$ an n-form with compact support on M and X a vector field on M. Prove that the integral of the Lie derivative of α along X vanishes:
$$\int_M \mathcal{L}_X \alpha = 0$$

(b) Prove a generalization of (a) for the case of M an oriented manifold with boundary (α and X are as above):
$$\int_M \mathcal{L}_X \alpha = \int_{\partial M} \iota_X \alpha$$
Here the boundary ∂M in the right hand side is equipped with the induced orientation from M.

(c) Use the above to compute $\int_{S^2} \iota_X \alpha$ where we see S^2 as the boundary of the closed 3-ball of unit radius $B \subset \mathbb{R}^3$, $\alpha = dx_1 \wedge dx_2 \wedge dx_3$ and $X = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}$.[1]

Problem 3. For G a group, its *abelianization*[2] $G^{\mathrm{ab}} = G/[G, G]$ is the abelian group defined as the quotient of G by the commutator subgroup $[G, G]$ – the normal subgroup of G generated by all elements of the form $aba^{-1}b^{-1}$ for $a, b \in G$.

(a) Prove that for the abelianization of the fundamental group of the surface of genus $g \geq 0$, there is a group isomorphism
$$\Big(\pi_1(\Sigma_g)\Big)^{\mathrm{ab}} \simeq \mathbb{Z}^{2g}$$

(b) Prove that for the abelianization of the fundamental group of the k-fold projective space (with $k \geq 1$), there is a group isomorphism
$$\Big(\pi_1(X_k)\Big)^{\mathrm{ab}} \simeq \mathbb{Z}^{k-1} \oplus \mathbb{Z}_2$$

(c) Compute $\Big(\pi_1(X)\Big)^{\mathrm{ab}}$ for $X = S^1 \vee S^1$ (a wedge of two circles).

[1] Recall that the volume of B is $\frac{4}{3}\pi$.

[2] Aside/motivation: one reason to be interested in abelianization is in the relation between the fundamental group π_1 and the first de Rham cohomology group H^1 of a connected manifold: if $[\pi_1(M)]^{\mathrm{ab}} \simeq \mathbb{Z}^m \oplus T$ with T a finite abelian group, then $H^1(M) \simeq \mathbb{R}^m$ with the same value of m.

Problem 4.

(a) Consider a manifold M equipped with a symplectic form ω.[3] A vector field X on a symplectic manifold (M, ω) is said to be *symplectic* if

$$\mathcal{L}_X \omega = 0$$

(i.e., ω is preserved by the flow of X). Also, a vector field on (M, ω) is said to be *Hamiltonian* if there exists a function $H \in C^\infty(M)$ such that

$$\iota_X \omega = dH$$

Prove that a Hamiltonian vector field is automatically symplectic.

(b) The converse of the statement of (a) is not always true (symplectic does not imply Hamiltonian), and here is an example. Let $M = S^1 \times \mathbb{R}$ be the cylinder with coordinates $\phi \in \mathbb{R}/2\pi\mathbb{Z}$ and $p \in \mathbb{R}$. Consider the symplectic form[4] $\omega = dp \wedge d\phi$ on M and a vector field $X = \frac{\partial}{\partial p}$. Prove that X is symplectic but not Hamiltonian.

Problem 5. For M a smooth manifold, prove that the cotangent bundle T^*M is orientable.

[3]Recall that ω is a symplectic form if it is a *closed* 2-form on M which is non-degenerate, i.e., induces a linear isomorphism $T_xM \xrightarrow{\simeq} T_x^*M$ for any point $x \in M$.

[4]Recall that on a circle, $d\phi$ is a closed but non-exact 1-form.