BASIC GEOMETRY AND TOPOLOGY HOMEWORK 1, DUE 8/21/2020

Problem 1. Consider the general linear group

$$GL_n(\mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) \mid \det A \neq 0\}$$

(a) Prove that $GL_n(\mathbb{R})$ is an open subset of $M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$.
(b) Prove that the map $f : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ defined by $f(A) = A^{-1}$ is continuous.[1]

Problem 2. Prove that the product topology on $\mathbb{R} \times \mathbb{R}$ (where both $\mathbb{R}$ factors are endowed with standard topology) agrees with the standard topology on $\mathbb{R}^2$.

Problem 3. Prove the continuity criterion for maps to a subspace: if X, Y are topological spaces and $A \subset Y$ a subset, then
(a) the inclusion $i : A \to Y$ is a continuous map;
(b) a map $f : X \to A$ is continuous if and only if the composition $X \xrightarrow{f} A \xrightarrow{i} Y$ is continuous.

Problem 4. Prove the continuity criterion for maps out of a quotient: if X, Y are topological spaces, $\sim$ an equivalence relation on X and $X/\sim$ the quotient space (with quotient topology), then:
(a) the quotient map $p : X \to X/\sim$ is continuous;
(b) a map $f : X/\sim \to Y$ is continuous if and only if the composition $X \xrightarrow{p} X/\sim \xrightarrow{f} Y$ is continuous.

Problem 5. British Rail metric on $\mathbb{R}^n$ is defined by

$$d(x,y) = \begin{cases} ||x|| + ||y|| & \text{if} x \neq y, \\ 0 & \text{if} x = y \end{cases} \quad (1)$$

(a) What do open balls $B_r^{BR}(x) = \{y \in \mathbb{R}^n \mid d(x,y) < r\}$ look like, depending on radius $r > 0$?
(b) What is the metric topology on $\mathbb{R}^n$ corresponding to the British Rail metric (1)? (I.e., the topology generated by subsets $B_r^{BR}(x)$ for $r > 0$, $x \in \mathbb{R}^n$.)

[1] Hint: compose f with the inclusion $GL_n(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$, show that the composition is continuous and infer that f itself is continuous by the continuity criterion for maps to a subspace.