BASIC GEOMETRY AND TOPOLOGY HOMEWORK 10, DUE 10/30/2020

I Prove the following properties of pullbacks of differential forms.

(a) If $F : M \to N$, $G : K \to M$ are two smooth maps and α is a p-form on N, then

$$(F \circ G)^* \alpha = G^*(F^* \alpha)$$

(b) For α, β two p-forms on N and $F : M \to N$ a smooth map, one has

$$F^*(\alpha + \beta) = F^*\alpha + F^*\beta$$

(c) For α a p-form on N, β a q-form on N and and $F : M \to N$ a smooth map, one has

$$F^*(\alpha \wedge \beta) = F^*\alpha \wedge F^*\beta$$

II Fix a smooth manifold M. Let Ξ^p be the space of skew-symmetric p-fold multilinear maps ζ from p-tuples of vector fields to functions on M such that

$$\zeta(X_1, \ldots, fX_i \ldots, X_p) = f\,\zeta(X_1, \ldots, X_i, \ldots, X_p)$$

for any $f \in C^\infty(M)$ – i.e., ζ is $C^\infty(M)$-linear in each argument. Construct an isomorphism (of vector spaces) between Ξ^p and the space $\Omega^p(M)$ of differential p-forms.

III **(Coordinate-free definition of the exterior derivative.)** Fix a p-form α on a manifold M. Consider a multilinear map A from $(p+1)$-tuples of vector fields $X_0, X_1, \ldots, X_p$ to smooth functions given by

$$\begin{aligned}(1)\quad A(X_0, \ldots, X_p) = &\sum_{i=0}^{p} (-1)^i X_i \Big(\alpha(X_0, \ldots, \widehat{X}_i, \ldots, X_p)\Big) + \\ &+ \sum_{0 \le i < j \le p} (-1)^{i+j} \alpha([X_i, X_j], X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_p)\end{aligned}$$

where the hat is the omission sign.

(a) Prove that A is skew-symmetric in $X_0, \ldots X_p$.

(b) Prove that A is $C^\infty(M)$-linear in each argument, i.e. $A(X_0, \ldots, fX_i, \ldots, X_p) = fA(X_0, \ldots, X_i, \ldots, X_p)$ for any $f \in C^\infty(M)$.[1] Thus, by Problem II, A corresponds to a $(p+1)$-form on M.

(c) Prove that $A = d\alpha$, using the definition of the exterior derivative on the right via local coordinates.

[1] It might be useful to first prove the following property of the Lie bracket: $[X, fY] = f[X, Y] + X(f)\, Y$, $[fX, Y] = f[X, Y] - X\, Y(f)$ for X, Y two vector fields and f a function.

IV Let $M = \mathbb{R}^3$. For f a function, $\alpha = a_1 dx_1 + a_2 dx_2 + a_3 dx_3$ a general 1-form and $\beta = b_1 dx_2 \wedge dx_3 + b_2 dx_3 \wedge dx_1 + b_3 dx_1 \wedge dx_2$ a general 2-form (here f, a_i, b_i are smooth functions of coordinates x_1, x_2, x_3), compute the exterior derivatives df, $d\alpha$, $d\beta$. Compare with formulas for the gradient of a function, curl of a vector field and divergence of a vector field.

V Consider a 2-form on an open set U in S^2 (the unit sphere in $\mathbb{R}^3$) given by

$$\omega = \sin\theta \, d\theta \wedge d\phi$$

where θ, ϕ are the spherical coordinates on S^2 and U is given by $\theta \in (0, \pi)$, $\phi \in (-\pi, \pi)$. Recall that the spherical coordinates (r, θ, ϕ) on $\mathbb{R}^3$ are related to Cartesian coordinates (x_1, x_2, x_3) by

$$x_1 = r \sin\theta \cos\phi, \quad x_2 = r \sin\theta \sin\phi, \quad x_3 = r \cos\theta$$

and the unit sphere is given by $r = 1$.

(a) Write ω in terms of the "stereographic coordinates" (u_1, u_2) where the stereographic chart map is $S^2 \backslash \{0, 0, 1\} \to \mathbb{R}^2$,

$$(x_1, x_2, x_3) \mapsto (u_1, u_2) = \frac{1}{1 - x_3}(x_1, x_2)$$

Also, write ω in terms of the opposite stereographic chart $S^2 \backslash \{0, 0, -1\} \to \mathbb{R}^2$ given by

$$(x_1, x_2, x_3) \mapsto (v_1, v_2) = \frac{1}{1 + x_3}(x_1, x_2)$$

(b) Using the previous, show that ω can be extended uniquely to a smooth 2-form on the entire S^2. (I.e. there exists a unique 2-form on S^2 which restricts to ω on $U \subset S^2$.)

(c) Let $\rho_t^i : \mathbb{R}^3 \to \mathbb{R}^3$, for $i = 1, 2, 3$, be the linear map of $\mathbb{R}^3$ into itself representing the rotation about x_i-axis by angle t. Note that diffeomorphisms ρ_t^i restrict to diffeomorphisms of S^2. Prove that these diffeomorphisms leave the 2-form ω invariant, in the sense that

$$(\rho_t^i)^* \omega = \omega$$

for any angle t and any $i = 1, 2, 3$.