# BASIC GEOMETRY AND TOPOLOGY HOMEWORK 11, DUE 11/6/2020

I Prove that the inner product of a vector field $X$ and a differential form $\alpha \in \Omega^p(M)$, defined via $(\iota_X\alpha)(X_1, \dots, X_{p-1}) = \alpha(X, X_1, \dots, X_{p-1})$ satisfies the Leibnitz identity

$$\iota_X(\alpha \wedge \beta) = \iota_X\alpha \wedge \beta + (-1)^p \alpha \wedge \iota_X\beta$$

Here $\alpha$ is a $p$-form and $\beta$ is a $q$-form.[1]

II Let $X$ and $Y$ be two vector fields on a manifold $M$ and $\alpha$ a $p$-form on $M$. Prove the following properties of the inner product and Lie derivative:
(a) $\iota_X\iota_Y\alpha = -\iota_Y\iota_X\alpha$.
(b) $\iota_X\mathcal{L}_Y\alpha - \mathcal{L}_Y\iota_X\alpha = \iota_{[X,Y]}\alpha$. Here $\mathcal{L}_Y$ is the Lie derivative along $Y$.
(c) $\mathcal{L}_X\mathcal{L}_Y\alpha - \mathcal{L}_Y\mathcal{L}_X\alpha = \mathcal{L}_{[X,Y]}\alpha$.

III Consider the vector field $X = x_1\frac{\partial}{\partial x_1} + \cdots + x_n\frac{\partial}{\partial x_n}$ on $\mathbb{R}^n$ (the "Euler vector field"). Show that if $f$ is a homogeneous polynomial of degree $k$ in coordinates $x_1, \dots, x_n$ and if $1 \leq i_1 < \dots < i_p \leq n$, then for the $p$-form $\alpha = f dx_{i_1} \wedge \cdots \wedge dx_{i_p}$, the Lie derivative along $X$ is:

$$\mathcal{L}_X\alpha = (k + p)\,\alpha$$

IV Let $M$ and $N$ be two compact smooth manifolds. For $p \geq 0$, construct a natural linear map to the de Rham cohomology of the product:

$$\Phi: \quad \bigoplus_{i=0}^{p} H^i(M) \otimes H^{p-i}(N) \to H^p(M \times N)$$

**Remark:** In fact (you don't have to prove this), $\Phi$ is an *isomorphism* of vector spaces and the fact that the cohomology of the product (r.h.s.) can be computed in terms of the cohomology of $M$ and $N$ (l.h.s.) is known as the Künneth formula.

V A *symplectic form* on a smooth $n$-manifold $M$ is a 2-form $\omega$ on $M$ such that
- $d\omega = 0$, i.e., $\omega$ is closed;
- $\omega$ is non-degenerate, i.e. for any $x \in M$, $\omega_x$ is a *non-degenerate* skew-symmetric bilinear form on the tangent space $T_xM$.[2]

[1] The following identity mentioned in class may be useful: $(\alpha \wedge \beta)(X_1, \dots, X_{p+q}) = \sum_{\sigma \in \mathrm{Sh}_{p,q}} \mathrm{sign}(\sigma) \cdot \alpha(X_{\sigma(1)}, \dots, X_{\sigma(p)})\,\beta(X_{\sigma(p+1)}, \dots, X_{\sigma(p+q)})$. Where the sum is over $(p,q)$-shuffles, i.e., permutations of $\{1, 2, \dots, p+q\}$ such that $\sigma(1) < \dots < \sigma(p)$ and $\sigma(p+1) < \dots < \sigma(p+q)$.

[2] I.e., one has skew-symmetry: $\omega_x(u, v) = -\omega_x(v, u)$ for any $u, v \in T_xM$ and non-degeneracy: $\omega_x(u, v) = 0$ for any $u \in T_xM$ implies $v = 0$.

(a) Show that in order to have a symplectic form, the manifold $M$ must necessarily have even dimension.

(b) Show that on $\mathbb{R}^{2n}$ with coordinates $x_1, \ldots, x_n, p_1, \ldots, p_n$, the 2-form

$$\omega = dx_1 \wedge dp_1 + dx_2 \wedge dp_2 + \cdots + dx_n \wedge dp_n$$

is a symplectic form.

(c) Consider $M = T^*N$ the cotangent bundle of a smooth manifold $N$. In a coordinate chart[3] $\pi^{-1}U$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ on $T^*N$ associated to a coordinate chart $U \subset N$ with coordinates $(x_1, \ldots, x_n)$ on the base $N$, define a 2-form locally as

$$\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n \tag{1}$$

- Prove that (1) defines a global[4] 2-form on $T^*N$ (by checking that expressions (1) written in terms of two coordinate charts on $T^*N$ agree on an overlap).
- Prove that the resulting global 2-form $\omega$ is a symplectic form on the cotangent bundle $T^*N$.

[3]Here $\pi : T^*N \to N$ is the bundle projection.

[4]As opposed to locally defined.