BASIC GEOMETRY AND TOPOLOGY HOMEWORK 3, DUE 9/4/2020

I Prove that the real projective space $\mathbb{RP}^n$ is a manifold for any $n \geq 1$ (Hint: adapt the construction for S^n we had in the class).

II Prove that the connected sum of two n-manifolds M, N is itself an n-manifold. In particular, for any point x on the identification sphere $S^{n-1} \subset M\#N$,[1] construct an open neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.

III (a) Assume that the finite cell complex $Z = X \cup Y$ is the union of cell subcomplexes X and Y intersecting over the subcomplex $W = X \cap Y$. Prove that the Euler characteristic satisfies $\chi(Z) = \chi(X) + \chi(Y) - \chi(W)$.[2]

(b) Prove that for M, N two connected compact 2-manifolds, the Euler characteristic of the connected sum satisfies $\chi(M\#N) = \chi(M) + \chi(N) - 2$. (Do not use the classification theorem for surfaces.)

IV (a) Prove that the "2-fold projective plane" $\mathbb{RP}^2\#\mathbb{RP}^2$ is homeomorphic to the Klein bottle.

(b) Which of the "standard surfaces" X_k, Σ_g is the surface corresponding to the word *abcbac* homeomorphic to? (Construct the homeomorphism using the moves we discussed in class.)

V (a) Prove that for X, Y two finite cell complexes, the Cartesian product $X \times Y$ also has a natural structure of a (finite) cell complex.

(b) Prove that the Euler characteristic of the product of finite cell complexes satisfies multiplicativity $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$.

[1] I.e., S^{n-1} is the image of the boundary sphere $\phi^{-1}(S^{n-1}) \subset M\backslash\phi^{-1}(B_1(0))$ or equivalently of the boundary sphere $\psi^{-1}(S^{n-1}) \subset N\backslash\psi^{-1}(B_1(0))$ in the quotient $M\#N = M\backslash\phi^{-1}(B_1(0)) \cup N\backslash\psi^{-1}(B_1(0))/\sim$.

[2] Recall that for X a finite cell complex, $\chi(X) := \sum_{k\geq 0}(-1)^k \#\{k\text{-cells in } X\}$.