BASIC GEOMETRY AND TOPOLOGY HOMEWORK 4, DUE 9/11/2020

I Assume that $X \subset \mathbb{R}^n$ is a "star shaped region," i.e. for any point $x \in X$, X contains all points on the straight line segment connecting x to the origin $0 \in \mathbb{R}^n$ (X is understood as endowed with subspace topology induced from $\mathbb{R}^n$). Show that X is simply connected.

II (Homotopy invariance of π_1.) Let (X, x_0) and (Y, y_0) be two pointed topological spaces. Assume that $\phi : X \to Y$ and $\psi : Y \to X$ are continuous maps such that

(i) $\phi(x_0) = y_0$, $\psi(y_0) = x_0$,

(ii) $\psi \circ \phi$ is homotopic to id_X via a homotopy F_t preserving the point x_0 for each $t \in [0, 1]$,

(iii) $\phi \circ \psi$ is homotopic to id_Y via a homotopy H_t preserving the point y_0 for each $t \in [0, 1]$.

(I.e. we are asking that (X, x_0) and (Y, y_0) are homotopy equivalent as pointed spaces.) Prove that then the induced homomorphism of fundamental groups

$$\phi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

is an isomorphism.

III For X a topological space and A a subspace, a "deformation retraction" is a family of continuous maps $f_t : X \to X$, for $t \in [0, 1]$, such that:

- $f_0 = \mathrm{id}_X$,
- $f_1(X) = A$,
- $f_t|_A = \mathrm{id}_A$ for any $t \in [0, 1]$,
- the family f_t is continuous in t, i.e., the map $F : X \times I \to X$ defined by $F(x, t) = f_t(x)$ is continuous.

In this case one says that A is a "deformation retract" of X (or "X retracts onto A").

(a) Prove that if $A \subset X$ is a deformation retract of X then A is homotopy equivalent to X.

(b) Prove that for X any topological space, $X \times I$ retracts onto $X \times \{0\}$.

(c) Prove that the punctured plane $\mathbb{C}\backslash\{0\}$ retracts onto the unit circle $S^1 \subset \mathbb{C}$.

(d) Prove that any star shaped region (cf. problem I) retracts onto the point $0 \in \mathbb{R}^n$.

IV (a) Consider the two-dimensional torus T as $\Sigma(aba^{-1}b^{-1})$ – the edge-gluing of a square. Let $\alpha \in \pi_1(T)$ be the homotopy class of the based loop in T corresponding to the edge of the square labelled by a; similarly, let $\beta \in \pi_1(T)$ be the homotpy class of the based loop in T corresponding to the edge labelled by b. Prove that α and β satisfy the following relation in $\pi_1(T)$:

$$\alpha\beta\alpha^{-1}\beta^{-1} = 1$$

(b) Similarly to (a), consider the Klein bottle $K = \Sigma(aba^{-1}b)$ and prove that the homotopy classes of loops α, β corresponding to the edges a, b satisfy the following relation in $\pi_1(K)$:

$$\alpha\beta\alpha^{-1}\beta = 1$$

V Give an example of a continuous map of pointed topological spaces $f : (X, x_0) \to (Y, y_0)$ such that the corresponding homomorphism of fundamental groups $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is neither injective nor surjective.