BASIC GEOMETRY AND TOPOLOGY HOMEWORK 7, DUE 10/9/2020

I Prove that if M is an m-dimensional smooth manifold and N is an n-dimensional smooth manifold, then the Cartesian product $M \times N$ has the structure of an $(m+n)$-dimensional smooth manifold.

II Consider the function $f : \mathbb{R}^2 \to R$ given by $f(x_1, x_2) = x_1 x_2$. Which $c \in \mathbb{R}$ in the codomain are the *regular*[1] values of f? (And hence $f^{-1}(c)$ is guaranteed to be a smooth manifold.)

III (a) Fix $n \geq 1$. Prove that the unitary group (a.k.a. the group of unitary matrices)

$$U(n) = \{A \in \mathrm{Mat}^{\mathbb{C}}_{n\times n} \mid A^\dagger A = \mathbf{1}\}$$

is a smooth manifold. What is its dimension?[2] Is it compact? Here $\mathrm{Mat}^{\mathbb{C}}_{n\times n}$ are square $n \times n$ matrices with complex entries, $A^\dagger = \bar{A}^T$ is the adjoint matrix (the conjugate-transpose), $\mathbf{1}$ is the unit matrix.

(b) Prove that the special linear group $SL(n, \mathbb{R}) = \{A \in \mathrm{Mat}_{n\times n} \mid \det A = 1\}$ is a smooth manifold of dimension $n^2 - 1$.[3]

IV (a) Find the tangent space to $SL(n, \mathbb{R})$ at $\mathbf{1}$ as $\ker DF_{\mathbf{1}} \subset \mathrm{Mat}_{n\times n}$ with $F : \mathrm{Mat}_{n\times n} \to \mathbb{R}$ given by $A \mapsto \det A$.

(b) Find the tangent space to $U(n)$ at $\mathbf{1}$ (as a subspace of $\mathrm{Mat}^{\mathbb{C}}_{n\times n}$).

---

[1] Recall that, given a map $F : \mathbb{R}^{m+n} \to \mathbb{R}^m$, a point $c \in \mathbb{R}^m$ is called a "regular value" of F if for each preimage $a \in F^{-1}(c)$, the differential $DF_a : \mathbb{R}^{m+n} \to \mathbb{R}^m$ is surjective.

[2] Warning: note that the dimension of $\mathrm{Mat}^{\mathbb{C}}_{n\times n}$ as a real vector space is $2n^2$, not n^2.

[3] A property of determinants you might find useful: for A an invertible matrix, one has $\det(A + H) = \det A \cdot (1 + \mathrm{tr}(A^{-1}H) + r(A, H))$ where $\lim_{H\to 0} r(A, H)/||H|| = 0$.

V Let S^n be the unit sphere in $\mathbb{R}^{n+1}$ with the atlas $(U^\pm, \phi^\pm)$ with

$$\begin{array}{ccc} U^\pm = S^n \backslash \{(0,\dots,0,\pm 1)\} & \xrightarrow{\phi^\pm} & \mathbb{R}^n \\ (x_0,\dots,x_n) & \mapsto & \frac{1}{1\mp x_n}(x_0,\dots,x_{n-1}) \end{array}$$

Denote $(u_1,\dots,u_n)$ the coordinates in chart (U^+,ϕ^+) and $(v_1,\dots,v_n)$ the coordinates in the chart (U^-,ϕ^-). Fix a point $a \in U^+ \cap U^-$.

(a) Let

$$\alpha = \sum_{i=1}^n \alpha_i (du_i)_a \quad \in T_a^* S^n \tag{1}$$

be a vector in the cotangent space[4] at a point $a \in S^n$, with α_i some fixed real coefficients. Express α defined by (1) in terms of the basis $\{(dv_i)_a\}$ in the cotangent space $T_a^* S^n$ (i.e. find the coefficients in terms of α_i's).

(b) Let

$$\xi = \sum_{i=1}^n \xi_i \left(\frac{\partial}{\partial u_i}\right)_a \quad \in T_a S^n \tag{2}$$

be a tangent vector to S^n at a, with ξ_i some coefficients. Express ξ in terms of the basis $\{\left(\frac{\partial}{\partial v_i}\right)_a\}$ in the tangent space $T_a S^n$.

[4] In the class we used the notation T_a^* for the cotangent space; here we are opting for the more explicit notation $T_a^* S^n$ to emphasize that it is the cotangent space to the sphere, not to the ambient $\mathbb{R}^{n+1}$.