BASIC GEOMETRY AND TOPOLOGY HOMEWORK 8, DUE 10/16/2020

I Prove the "regular value level set theorem" for manifolds formulated in class: if $F : M \to N$ is a smooth map of smooth manifolds and $c \in N$ a point such that for any $a \in F^{-1}(c)$, the derivative $DF_a : T_aM \to T_{F(a)}N$ is surjective, then $F^{-1}(c)$ is a submanifold of M. Denoting $\iota : F^{-1}(c) \to M$ the inclusion map, prove that $\mathrm{im}(D\iota_a) = \ker(DF_a)$ for any $a \in M$.[1]

II One calls a smooth map between smooth manifolds $F : M \to N$ an *immersion* if DF_a is injective for each $a \in M$. One calls F a *submersion* if DF_a is surjective for each $a \in M$.[2]
(a) Is the map $\mathbb{R} \to \mathbb{R}^2$ given by $t \mapsto (t^2, t^3)$ an immersion?
(b) Is the map $\mathbb{R} \to \mathbb{R}$ given by $t \to t^3$ a submersion?
(c) Is the map $\mathbb{R}^2 \backslash \{0\} \to S^1$ given by $x \mapsto \frac{x}{||x||}$ (with $x \in \mathbb{R}^2 \backslash \{0\}$) a submersion?

III Denote $\iota : S^2 \to \mathbb{R}^3$ the inclusion of S^2 as a unit sphere in $\mathbb{R}^3$. Let $f = \iota^* x_3$ the pullback of the coordinate function x_3 on $\mathbb{R}^3$ to S^2 by the inclusion. Compute the derivative df_a (for a general point a) in a stereographic chart on S^2,

$$\begin{array}{rcc} S^2 \backslash (0,0,1) & \to & \mathbb{R}^2 \\ (x_1, x_2, x_3) & \mapsto & (u_1, u_2) = \frac{1}{1-x_3}(x_1, x_2) \end{array}$$

IV Let $f : M \to \mathbb{R}$ be a smooth function on a smooth manifold M and fix a point $a \in M$. Explain how to identify the derivative $df_a \in T_a^* M$ of f as a function and the derivative $Df_a : T_aM \to T_{f(a)}\mathbb{R}$ of f as a map between manifolds?

V Consider a one-parameter group of diffeomorphisms of $\mathbb{R}^2$ given by $\phi_t : (x_1, x_2) \mapsto (\cos(t)x_1 - \sin(t)x_2, \sin(t)x_1 + \cos(t)x_2)$.
(a) Find the vector field X corresponding to ϕ_t (via differentiation at $t = 0$).
(b) Calculate $X(f)$ where $f = (x_1)^2 + (x_2)^2$.

[1] You can use the case $F : \mathbb{R}^{m+n} \to \mathbb{R}^m$ discussed in the class as a starting point.

[2] Note that the regular value level set theorem implies that each fiber of a submersion $F : M \to N$ (i.e. preimage $F^{-1}(c)$ of any $c \in N$) is a submanifold of M.