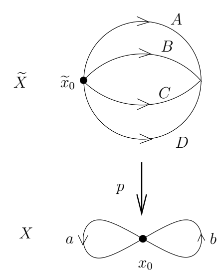
MIDTERM EXAM, DUE 9/23/2020 AT 11AM

Problem 1. Prove that any continuous map $f : \mathbb{RP}^2 \to S^1$ is homotopic to the constant map f_0 sending each point of $\mathbb{RP}^2$ to the point $1 \in S^1$.[1]

Problem 2. Consider the covering



Here $\widetilde{X}$ and X are understood as graphs (1-dimensional CW complexes); $\widetilde{X}$ is a graph with two vertices and four edges and X is a graph with one vertex and two edges. The covering map homeomorphically identifies the edges of $\widetilde{X}$ with the edges of X according to

$$p: \qquad A \to a,\ B \to \bar{a},\ C \to b,\ D \to \bar{b}$$

where the overline means "traverse the edge in the opposite direction." Describe explicitly[2] the subgroup $H = p_* \pi_1(\widetilde{X}, \widetilde{x}_0)$ in $G = \pi_1(X, x_0) = \langle \alpha, \beta \rangle$. Here $\alpha = [a]$, $\beta = [b]$ are the homotopy classes of loops a, b.

Problem 3. Consider the "line with two origins" – the topological space

$$X = \mathbb{R} \sqcup \mathbb{R} / \sim$$

with the equivalence relation $(x, 1) \sim (x, 2)$ for any $x \in \mathbb{R} \backslash \{0\}$. Here we understand the disjoint union $\mathbb{R} \sqcup \mathbb{R}$ as $\mathbb{R} \times \{1, 2\}$. Prove that X satisfies the axioms of a topological 1-manifold, except that it fails the Hausdorff property.

---

[1] Hint: it may be useful to first prove that f must have a lifting $\tilde{f} : \mathbb{RP}^2 \to \mathbb{R}$ along the standard covering map $p : \mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$ (i.e. such that $p \circ \tilde{f} = f$). Then prove that $\tilde{f}$ is homotopic to a constant map $\mathbb{RP}^2 \to \mathbb{R}$, mapping everything to zero. Use this homotopy to construct a homotopy between the original map f and the constant map to S^1.

[2] I.e. describe H as a subgroup of G generated by certain explicit elements – words in $\alpha^{\pm 1}, \beta^{\pm 1}$.

Problem 4. Construct an explicit isomorphism[3] between the fundamental group of the Klein bottle presented as $\langle a, b | aba^{-1}b = 1\rangle$ and the fundamental group of $\mathbb{RP}^2 \# \mathbb{RP}^2$ presented as $\langle c, d | c^2 d^2 = 1\rangle$.[4]

Problem 5. Prove that the Grassmanian $G_k(\mathbb{R}^n)$ – the space of k-dimensional subspaces in $\mathbb{R}^n$ (with $0 \leq k \leq n$) – is a *compact* topological space.[5]

[3]I.e. give explicitly the value of the isomorphism on the generators.

[4]Hint: it might be useful to inspect in detail the homeomorphism $K \approx \mathbb{RP}^2 \# \mathbb{RP}^2$. – One can track where do the curves represented by the sides of the square out of which K is glued go under this homeomorphism.

[5]Recall that one has a surjective map $p : V_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n)$ from the Stiefel manifold $V_k(\mathbb{R}^n) = \{(v_1, \ldots, v_k) \,|\, v_i \in \mathbb{R}^n,\, v_i \cdot v_j = \delta_{ij}\}$ where p maps an orthonormal k-tuple of vectors $(v_1, \ldots, v_k)$ to the subspace $\mathrm{Span}\{v_1, \ldots, v_k\} \subset \mathbb{R}^n$.