Symplectic geometry

Ref: Ana Cannas da Silva "Lectures on symplectic geometry"

Symplectic linear algebra

Let V - m-dim. vector space over $\mathbb{R}$, $B: V \times V \longrightarrow \mathbb{R}$ a

skew-symmetric bilinear form, i.e. $B(u,v) = -B(v,u)$ $\forall u,v \in V$

Thm (standard form for skew-symmetric bilinear forms)

For B a s-s. b.f. on V, there exists a basis $h_1, \dots, h_k; e_1, \dots, e_n, f_1, \dots, f_n$ on V s.t.

- $B(h_i, v) = 0$ $\forall v \in V$, any i
- $B(e_i, e_j) = 0 = B(f_i, f_j)$ $\forall i,j$
- $B(e_i, f_j) = \delta_{ij}$ $\forall i,j$

Rem a) such a basis is not unique (though called a "canonical basis")

b) in matrix notation, $B(u,v) = (\cdots u \cdots) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{Id} \\ 0 & -\mathrm{Id} & 0 \end{pmatrix} \begin{pmatrix} \vdots \\ v \\ \vdots \end{pmatrix}$

Proof [skew-sym version of Gram-Schmidt]

Let $H = \{u \in V \mid B(u,v) = 0 \ \forall v \in V\}$ choose a basis $h_1, \dots, h_k$ in H

and choose a complement W to H in V, i.e., $V = H \oplus W$

- choose $e_1 \in W$, $e_1 \neq 0$. Then $\exists f_1 \in W$ s.t. $B(e_1, f_1) \neq 0$. Normalize f_1 so that $B(e_1, f_1) = 1$.

let $W_1 = \mathrm{span}\{e_1, f_1\}$, $W_1' = \{w \in W \mid B(w,v) = 0 \ \forall v \in W_1\}$

- one has $W_1 \cap W_1' = \{0\}$ (indeed, if $v = a e_1 + b f_1 \in W_1 \cap W_1'$

then $\underbrace{B(v, e_1)}_{-b} = 0$, $\underbrace{B(v, f_1)}_{a} = 0 \Rightarrow v = 0$)

- one has $W = W_1 \oplus W_1'$

(indeed: if $w \in W$, $B(w, e_1) =: -b$, $B(w, f_1) =: a$, then $w = \underbrace{(a e_1 + b f_1)}_{\in W_1} + \underbrace{(w - a e_1 - b f_1)}_{\in W_1'}$)

Repeat:

- choose $e_2 \in W_1'$, $e_2 \neq 0$, $f_2 \in W_1'$ s.t. $B(e_2, f_2) = 1$, let $W_2 = \mathrm{span}(e_2, f_2)$, ...

Eventually: $V = H \oplus W_1 \oplus W_2 \oplus \dots \oplus W_n$ - all summand are $\perp$ w.r.t. B,

$W_i = \mathrm{span}\{e_i, f_i\}$, $B(e_i, f_i) = 1$

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$k = \dim H$ indep. of the choice of basis

$2n = \dim V - \dim H$ - "rank" of B.

For V, B - s.s.b.f., let $B^{\#} : V \to V^*$ be the linear map defined by $B^{\#}(v)(u) := B(v,u)$

- $\ker B^{\#} = H$ above

def If $B^{\#} : V \to V^*$ is bijective (i.e. $H=0$), then the s.s.b.f. B is called "symplectic" (or "nondegenerate")

B is then called a <u>linear symplectic structure</u> on V and (V,B) is called a <u>symplectic vector space.</u>

For (V,B) symplectic,

- $B^{\#} : V \xrightarrow{\sim} V^*$ is a lin. isomorphism
- by normal form thm, $\dim H = 0 \Rightarrow \dim V = 2n$ is even
- V has a basis $e_1, \dots, e_n, f_1, \dots, f_n$ with $B(e_i, f_j) = \delta_{ij}$, $B(e_i, e_j) = 0 = B(f_i, f_j)$
  - a "symplectic basis" for (V,B).

One has $B(u,v) = (\cdots u \cdots) \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix} \begin{pmatrix} \vdots \\ v \\ \vdots \end{pmatrix}$

For (V,B) a symp. v. sp. and $Y \subset V$ a linear subspace,

$Y^{\perp} := \{ v \in V \mid B(v,u) = 0 \quad \forall u \in Y \}$ is called the "symplectic orthogonal" of Y.

<u>Properties of symp. orthogonals</u>

- $\dim Y + \dim Y^{\perp} = \dim V$.
- $(Y^{\perp})^{\perp} = Y$
- if $\underbrace{Y \subset Y'}_{\text{subspaces}} \subset V$, then $(Y')^{\perp} \subset Y^{\perp}$

Distinguished classes of subspaces of (V,B)

a subspace $Y \subset V$ is called

a) <u>symplectic</u> if $B|_Y$ is non-degenerate $\Leftrightarrow$ $Y \cap Y^\perp = \{0\} \Leftrightarrow V = Y \oplus Y^\perp$ — e.g. $Y = \mathrm{span}\{e_1, f_1\}$

b) <u>isotropic</u> if $B|_Y = 0$ (i.e. $Y \subset Y^\perp$) — e.g. $Y = \mathrm{span}\{e_1, e_2\}$

c) <u>coisotropic</u> if $Y^\perp \subset Y$ — e.g. $Y = \mathrm{span}\{e_1, e_2, f_1\}$ for $n=2$ ($Y^\perp = \mathrm{span}\{e_2\}$)

d) <u>Lagrangian</u> if $Y^\perp = Y$
(i.e. if Y is isotropic and coisotropic)

<u>Ex.</u> (prototype of a symplectic v.sp.):

$(\mathbb{R}^{2n}, B_0)$, with B_0 such that

$$e_i = (\underbrace{0 \cdots \overset{\uparrow\, i^{th}\text{ place}}{1} \cdots 0}_{n}; \underbrace{0 \cdots 0}_{n})$$

$$f_i = (\underbrace{0 \cdots 0}_{n}; \underbrace{0 \cdots \overset{(n+i)^{th}\text{ place}}{1} \cdots 0}_{n})$$

– symp. basis

<u>def</u> A <u>symplectomorphism</u> φ between symp. v. spaces (V,B) and (V',B') is a linear iso $\varphi : V \longrightarrow V'$ s.t. $\varphi^* B' = B$ (i.e. $B(u,v) = \underbrace{B'(\varphi(u), \varphi(v))}_{=: (\varphi^* B')(u,v)}$)

If a symplectomorphism exists, (V,B) and (V',B') are said to be symplectomorphic.

<u>Rem</u>
- being symplectomorphic is an equiv. relation on symp. v. spaces
- every symp. v.sp. (V,B), $\dim V = 2n$ is symplectomorphic to the standard one, $(\mathbb{R}^{2n}, B_0)$, by

$$V \longrightarrow \mathbb{R}^{2n}$$

symp. basis for B $\left\{ \begin{array}{l} e_i \longmapsto e_i^0 \\ f_i \longmapsto f_i^0 \end{array} \right\}$ stand. symp. basis for B_0

<u>Thus</u>: there is a single equivalence class in each even dimension.

Symplectic manifolds

def A symplectic form on a (smooth) manifold M is a 2-form $\omega \in \Omega^2(M)$ s.t.

a) $d\omega = 0$

b) for each $p \in M$, $\omega_p : T_pM \times T_pM \xrightarrow[\text{skew-symmetric}]{\text{bilinear}} \mathbb{R}$ is symplectic (nondegenerate)

Note: if ω is symplectic, then $\dim M = \dim T_pM$ is even

def A symplectic manifold is a pair (M, ω) with M a (smooth) manifold and ω a symplectic form.

Ex: $M = \mathbb{R}^{2n}$ with linear coords $x_1, \dots, x_n, y_1, \dots, y_n$.

$\omega_0 = \sum_i dx_i \wedge dy_i$ is a symplectic form;

the set $\left(\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_n}\right)_p, \left(\frac{\partial}{\partial y_1}\right)_p, \dots, \left(\frac{\partial}{\partial y_n}\right)_p \right)$ is a symp. basis for T_pM

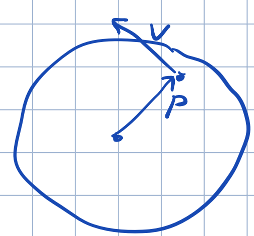
Ex: $M = \mathbb{C}^n$ with linear cx coords $z_1, \dots, z_n$.

$\omega_0 = \frac{i}{2} \sum_k dz_k \wedge d\bar{z}_k$ is a symp. form. - It equals the stand. symp. form on $\mathbb{R}^{2n}$ under the identification $\mathbb{C}^n \simeq \mathbb{R}^{2n}$

$z_k = x_k + i y_k$

Ex: $M = S^2$ seen as a unit sphere in $\mathbb{R}^3$

$T_pS^2 \cong \{ v \in \mathbb{R}^3 \mid \langle v, p \rangle = 0 \}$



Symp. form:

$\omega_p(u, v) := \langle p, u \times v \rangle$ for $u, v \in T_pS^2$

$\omega \in \Omega^2(S^2)$ - top-degree form $\Rightarrow d\omega = 0$

ω_p is nondegenerate: $\omega_p(\underset{\substack{\neq \\ 0}}{u}, v = u \times p) \neq 0$.

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def Let $(M_1, \omega_1), (M_2, \omega_2)$ be $2n$-dimensional symp. manifolds and $\varphi: M_1 \to M_2$ a diffeomorphism. Then φ is a symplectomorphism if $\varphi^* \omega_2 = \omega_1$.

We are interested in classifying symp. mflds up to symplectomorphism.

It turns out, locally a symp. mfld is symp-morphic to $\mathbb{R}^{2n}, \omega_0$!

Darboux theorem

Let (M,ω) be $2n$-dimensional symp. mfd and $p \in M$ any point.

Then there is a coord. chart $(\mathcal{U}, x_1,\dots,x_n, y_1,\dots,y_n)$ centered at p s.t. on $\mathcal{U}$:

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$$

The chart satisfying is called a Darboux chart.

Thus: the model for a local piece of a $2n$-dim. symp. mfd is $M = \mathbb{R}^{2n}$ with lin. coords $x_1 \dots x_n, y_1 \dots y_n$ and with symp. form $\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$.

Canonical symplectic form on a cotangent bundle

for X an n-mfd, consider the cotangent bundle T^*X

if $(\mathcal{U}, x_1,\dots,x_n)$ - coord. chart on M, then $\{(dx_i)_x\}_{i=1\dots n}$ - basis in T_x^*X

$x_i : \mathcal{U} \to \mathbb{R}$

$\to \xi \in T_x^*X$ can be written as $\xi = \sum_i \underset{\substack{\uparrow \\ \text{coefficients}}}{\xi_i} (dx_i)_x$

$\to$ $T^*\mathcal{U} \longrightarrow \mathbb{R}^{2n}$ - coord. chart on T^*X

$(x,\xi) \longmapsto \underbrace{(x_1,\dots,x_n,\xi_1,\dots,\xi_n)}_{\text{cotangent coordinates}}$

<u>def</u> The "tautological form" (or Liouville 1-form) $\alpha \in \Omega^1(T^*X)$ is the 1-form given in cotangent coords by $\alpha := \sum_i \xi_i \, dx_i$. $(*)$

The canonical symplectic form on T^*X is $\omega := -d\alpha = \sum_i dx_i \wedge d\xi_i \in \Omega^2(T^*X)$

<u>Rem</u> local definition $(*)$ is <u>consistent</u>: if $\left(T^*\mathcal{U}, \begin{smallmatrix} x_1 \dots x_n, \\ \xi_1 \dots \xi_n \end{smallmatrix}\right), \left(T^*\mathcal{U}', \begin{smallmatrix} x_1' \dots x_n' \\ \xi_1' \dots \xi_n' \end{smallmatrix}\right)$ two cotangent charts,

then on $T^*(\mathcal{U} \cap \mathcal{U}')$, one has

$$\alpha' = \sum_i \xi_i' \, dx_i' = \sum_i \underbrace{\left(\sum_j \xi_j \left(\frac{\partial x_j}{\partial x_i'}\right)\right)}_{\xi_i'} \underbrace{\left(\sum_k \left(\frac{\partial x_i'}{\partial x_k}\right) dx_k\right)}_{dx_i'} = \sum_i \xi_i \, dx_i = \alpha .$$