Splitting principle

Thm For $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ a rk=n cx. v.bun. there exists a space X and a map $f: X \to M$

s.t. a) $\begin{matrix} f^*E \\ \downarrow \\ X \end{matrix}$ splits into line bundles, $f^*E = L_1 \oplus \cdots \oplus L_n$

b) $f^*: H^\bullet(M, \mathbb{Z}) \to H^\bullet(X, \mathbb{Z})$ is injective.

Idea: $n = 1$ - nothing to prove, $X = M$, $f = id$

- $n > 1$: choose a fiber Herm. metric in E;

Set $X_1 := PE$, $(PE)_x = \{\text{lines in } E_x\}$

$$\begin{matrix} PE \\ \downarrow f_1 \\ M \end{matrix}$$

projectivization of fibers

$\begin{matrix} f_1^*E \\ \downarrow \\ PE \end{matrix}$ contains a tautological line subbundle L_1, $(L_1)_{(x,\ell)} = \ell$ ← line in E_x

so, $\begin{matrix} f_1^*E = L_1 \oplus \underbrace{\boxed{L_1^\perp}}_{\text{rk}=n-1 \text{ v.bun.}} =: E_1 \\ \downarrow \\ PE \end{matrix}$

- $f_1^*: H^\bullet(M, \mathbb{Z}) \to H^\bullet(PE, \mathbb{Z})$ is injective

(from Leray-Hirsch theorem $\Rightarrow H^\bullet(PE) \simeq H^\bullet(M) \otimes H^\bullet(\mathbb{C}P^{n-1})$)
- thm 4D.1, Hatcher AT

- Iterate:

$$\begin{tikzcd} E \arrow[d] & L_1 \oplus E_1 \arrow[l] \arrow[d] & f_2^*L_1 \oplus L_2 \oplus E_2 \arrow[l] \arrow[d] & \cdots \arrow[l] & (\ )^*L \oplus \cdots \oplus L_n \arrow[l] \arrow[d] & \text{splitting into line bundles} \arrow[l] \\ M & PE \arrow[l, "f_1"] & PE_1 \arrow[l, "f_2"] & \cdots \arrow[l] & X \arrow[l, "f_{n-1}"] & \\ & & f & & & \end{tikzcd}$$

$f^* = f_{n-1}^* \cdots f_1^*: H(M) \to H(X)$ injective.

$X = Fl(E) = \{(x, \text{orthogonal splittings of the fiber into lines } E_x = \ell_1 \oplus \cdots \oplus \ell_n)\}$

LAST TIME

Chern-Weil homomorphism

a) for $\begin{array}{c} P \circlearrowleft G \\ \downarrow{\scriptstyle \pi} \\ M \end{array}$ a G-bundle, we have an algebra homom.

$$\Phi : \underbrace{\mathrm{Inv}(\mathfrak{g})}_{(S^\bullet \mathfrak{g}^*)^G \text{ - inv. poly. on } \mathfrak{g}} \longrightarrow H^\bullet(M; \overset{\mathbb{R} \text{ or } \mathbb{C}}{\Bbbk})$$

$$p \longmapsto [p(\overset{\text{curvature of any connection } A.}{F_A})]$$

$$\overset{\shortparallel}{(\pi^*)^{-1} p(F_A)} \quad , \quad \pi^* : \Omega^\bullet(M) \to \Omega^\bullet(P)^{\text{basic}}$$

naturality:

- Ψ is compatible with pullbacks of G-bundles: $\Psi(p, f^* P) = f^* \Psi(p, P)$

b) $\hat{\Psi} : \mathrm{Inv}(\mathfrak{g}) \longrightarrow H^\bullet(BG; \Bbbk)$

- For G cpt, $\hat{\Psi}$ is an iso.

* * *

- For $G = GL(n, \mathbb{C})$, $\mathrm{Inv}(\mathfrak{g}) = \mathbb{C}[\overset{\text{elem. sym. poly in eigenvalues } \lambda_1 \dots \lambda_n \text{ of } X \in \mathfrak{g}}{\sigma_1, \dots, \sigma_n}]$

$s_j = \sum \lambda_i^j = \mathrm{tr}\, X^j$

$\sigma_1 = \mathrm{tr}\, X$, $\sigma_n = \det X$

$$\underset{X \mapsto \mathrm{tr} X}{\Phi(\sigma_1)} \left(\begin{array}{c} P \circlearrowleft G \\ \downarrow \\ M \end{array} \right) = [\mathrm{tr}\, F_A] = -2\pi i \, c_1^{\mathbb{C}}(P) \implies \underset{\substack{\uparrow \\ \text{image of } c_1(P) \text{ under } H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{C})}}{c_1^{\mathbb{C}}}(P) = \left[-\frac{1}{2\pi i} \mathrm{tr}\, F_A \right] \in H^2_{\text{de Rham}}(M)$$

Cor: $\displaystyle\int_{\underset{\nwarrow \text{closed 2d surface}}{\Sigma \subset M}} -\frac{1}{2\pi i} \mathrm{tr}\, F_A \in \mathbb{Z}$ for any $A \in \mathrm{Conn}(P)$

- [switching to vector bundles]

for $\begin{array}{c} E = L_1 \oplus \dots \oplus L_n \\ \downarrow \\ M \end{array}$ a cx. v. bun splitting into line bundles,

$$\overset{\text{total Chern class}}{c(E)} = \prod_j (1 + c_1(L_j)) \xrightarrow[\mathbb{Z} \to \mathbb{C}]{\text{coeffs}} \left[\prod_j \left(1 - \frac{1}{2\pi i} F_{\underset{\substack{\nearrow \\ \text{v.bun. connection in } \begin{array}{c} L_j \\ \downarrow \\ M \end{array}}}{A_j}} \right) \right] =$$

$$= \left[\det \left(1 - \frac{1}{2\pi i} F_{\underset{\substack{\uparrow \\ A = A_1 \oplus \dots \oplus A_n \text{ - conn. in } \begin{array}{c} E \\ \downarrow \\ M \end{array}}}{A}} \right) \right] \in H^\bullet(M, \mathbb{C})$$

Locally $(A_i)_\alpha \in \Omega^1(U_\alpha; \overset{\substack{gl(1,\mathbb{C}) \\ \downarrow}}{\mathbb{C}})$,

$A_\alpha = \begin{pmatrix} (A_1)_\alpha & & 0 \\ & \ddots & \\ 0 & & (A_n)_\alpha \end{pmatrix} \in \Omega^1(U_\alpha; gl(n, \mathbb{C}))$

- for $\begin{array}{c} E \\ \downarrow \\ M \end{array}$ a general $rk = n$ cx. v. bun,

by splitting principle, $\exists f : \underset{\substack{\shortparallel \\ \mathrm{Flag}(E)}}{N} \longrightarrow M$ s.t. $\begin{array}{c} f^* E = L_1 \oplus \dots \oplus L_n \\ \downarrow \\ N \end{array}$

and $f^* : H^\bullet(M) \to H^\bullet(N)$ injective.

Then: $c^{\mathbb{C}}(f^*E) = \left[\det\left(1-\frac{1}{2\pi i}F_{f^*\boxed{\mathcal{A}}}\right)\right] \in H^\bullet(N,\mathbb{C})$ ← Conn. on E

// naturality $\qquad\qquad$ $\shortparallel$

$f^* c^{\mathbb{C}}(E)$ $\qquad$ $f^*\left[\det\left(1-\frac{1}{2\pi i}F_{\mathcal{A}}\right)\right]$

$$\Rightarrow \boxed{c^{\mathbb{C}}(E) = \left[\det\left(1-\frac{1}{2\pi i}F_{\mathcal{A}}\right)\right]} \quad (**) \text{ for any } \begin{matrix}E\\ \downarrow\\ M\end{matrix} \text{ cx. v. bun.}$$

$$=\left[1 + \underbrace{\left(\frac{-1}{2\pi i}\operatorname{tr}F_{\mathcal{A}}\right)}_{c_1} + \underbrace{\left(\frac{-1}{2\pi i}\right)^2\frac{1}{2}\underbrace{\Big(\underbrace{(\operatorname{tr}F_{\mathcal{A}})^2}_{s_1^2} - \underbrace{\operatorname{tr}F_{\mathcal{A}}\wedge F_{\mathcal{A}}}_{s_2}\Big)}_{\sigma_2}}_{c_2} + \dots\right]$$

$$= \sum_{j=0}^{n} \sigma_j\left(\text{eigenvalues of } \tfrac{-1}{2\pi i}F_{\mathcal{A}}\right) = \sum_{j=0}^{n}\Gamma_j\left(s_\ell\left(\text{eigenvalues of } \tfrac{-1}{2\pi i}F_{\mathcal{A}}\right)\right) = \sum_{j=0}^{n}\Gamma_j\Big(\big\{s_\ell = \left(\tfrac{-1}{2\pi i}\right)^\ell \operatorname{tr}\underbrace{F_{\mathcal{A}}\wedge\dots\wedge F_{\mathcal{A}}}_{\ell}\big\}_{\ell\le j}\Big)$$

↑ polynomials expressing σ_j in terms of s_ℓ in $(\mathbb{Q}[x_1,\dots,x_n])^{S_n}$ $\Gamma_j(s_1,\dots,s_j)$

$\Gamma_0 = 1$

$\Gamma_1 = s_1$

$\Gamma_2 = \frac{1}{2}(s_1^2 - s_2)$

$\Gamma_3 = \frac{1}{3}s_3 + \frac{1}{6}s_1^3 - \frac{1}{2}s_1 s_2$

$\dots$

- in terms of $\hat{\Psi}: \mathrm{Inv}(\mathfrak{g}) \to H^\bullet(BG,\mathbb{C})$: $\qquad$ univ. Chern class ↓ $c_j^{\mathbb{C}} = \hat{\Psi}\left(\sigma_j\left(\frac{-1}{2\pi i}\cdot\text{eigenvalues}\right)\right) = \left(\frac{-1}{2\pi i}\right)^j\hat{\Psi}(\sigma_j)$

- Same formula (**) holds for principal $GL(n,\mathbb{C})$-bundles. (or $U(n)$ -)

$E \longmapsto P = FE$ $\qquad$ and $\quad P \longmapsto E = P \times_G \mathbb{R}^n$

$\mathcal{A}^E \longmapsto \mathcal{A}^P$ $\qquad\qquad\quad$ $\mathcal{A}^P \longmapsto \mathcal{A}^E$

$\mathrm{hol}_{\gamma:x\to x}\mathcal{A}^E = \mathrm{hol}_\gamma \mathcal{A}^P$

↑ closed loop $\quad$ $\overset{\text{in}}{\in}\ \mathrm{Iso}(E_x) \simeq G = GL(n,\mathbb{C})$

for $\begin{matrix}P \circlearrowleft G\\ \downarrow\\ M\end{matrix}$ with a connection $\mathcal{A}^P$ and $G \xrightarrow{\rho} GL(V)$ a repr.,

one has an induced connection on $E = P \times_G V$ with

$$\begin{tikzcd}\nabla^\rho = d + \rho(\mathcal{A}^P): & \Omega^\bullet(P,V)^{\text{basic}} \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \Omega^{\bullet+1}(P,V)^{\text{basic}} \arrow[d, phantom, "\cong" rotate=90] \\ & \Omega^\bullet(M,E) \arrow[r, "\boxed{\nabla^E}"'] & \Omega^{\bullet+1}(M,E)\end{tikzcd}$$

Rem For $\begin{matrix}P\\ \downarrow\\ M\end{matrix}$ an $SU(n)$-bundle, $\quad$ locally $F_{\mathcal{A}} \in \Omega^2(M, \mathfrak{su}(n)) \Rightarrow \operatorname{tr}F_{\mathcal{A}} = 0$

↑ traceless skew-self-adjoint matrices

$\Rightarrow c_1^{\mathbb{C}} = 0\,,\quad c_2^{\mathbb{C}} = \left[\frac{1}{8\pi^2}\operatorname{tr}F_{\mathcal{A}}\wedge F_{\mathcal{A}}\right]$

↑ in fact $H^\bullet(BSU(n),\mathbb{Z}) = \mathbb{Z}[c_2, c_3, \dots, c_n]$ no c_1

thus, $\displaystyle\int_{X^4\subset M} \frac{1}{8\pi^2}\operatorname{tr}F_{\mathcal{A}}\wedge F_{\mathcal{A}} \in \mathbb{Z}$ for any connection $\mathcal{A}$.

↑ 4-cycle

Pontrjagin classes

Let $\begin{tikzcd} E \arrow[d] \\ M \end{tikzcd}$ a rk $= n$ real v. bun. ; equip it with fiber metric.

bundle of o/n frames

$$\begin{tikzcd} P = F_O(E) \circlearrowleft O(n) \arrow[d] \\ M \end{tikzcd}$$

$$\alpha \in \Omega^1(P; o(n)) \quad , \quad \mathcal{F}_\alpha \in \Omega^2(P; o(n))^{\text{basic}}$$

skew-sym matrices $X^T = -X$

$$\updownarrow$$

$$F_\alpha \in \Omega^2(M, ad(P)) \quad , \text{ locally, } (F_\alpha)_\alpha \in \Omega^2(M, o(n))$$

- $\operatorname{tr} \mathcal{F}_\alpha^{\wedge j} = 0$ for j odd , $\{\text{eigenvalues of } \mathcal{F}_\alpha\} = \{-\text{eigenvalues of } \mathcal{F}_\alpha\}$
  $\Rightarrow c^{\mathbb{C}}_{odd}(E_{\mathbb{C}}) = [c_{odd}(\mathcal{F}_\alpha)] = 0$

$$p^{\mathbb{R}}_j(E) = (-1)^j c^{\mathbb{C}}_{2j}(E_{\mathbb{C}}) = \left[\frac{1}{(2\pi)^{2j}} \Gamma_{2j}\left(s_1 = 0, s_2 = \operatorname{tr}(F_\alpha)^{\wedge 2}, s_3 = 0, s_4 = \operatorname{tr}(\mathcal{F}_\alpha)^{\wedge 4}, \dots \right)\right]$$

$$\in H^{4j}_{\text{de Rham}}(M, \mathbb{R})$$

So: $p^{\mathbb{R}}_1(E) = -\frac{1}{8\pi^2} \operatorname{tr}(F_\alpha)^{\wedge 2}$

$p^{\mathbb{R}}_2(E) = \frac{1}{128\pi^4} \left((\operatorname{tr} F_\alpha^{\wedge 2})^2 - 2 \operatorname{tr} F_\alpha^{\wedge 4} \right)$

-- - - - - . . .

Euler class

- for $X \in O(2n)$ a skew-symmetric matrix, the Pfaffian is:

$$Pf(X) \overset{\text{def}}{=} \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} (-1)^\sigma \prod_{i=1}^{n} X_{\sigma_{2i-1} \sigma_{2i}} = \sum_{\text{partitions}} \pm X_{i_1 j_1} \cdots X_{i_n j_n}$$

$\pm$: sign of permutation

partitions: $\{1, \dots, 2n\} = \{i_1, j_1\} \cup \dots \cup \{i_n, j_n\}$ without regard to order

E.g. $Pf\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a$

properties:
- $Pf(X)^2 = \det(X)$
- $Pf(YXY^T) = \det(Y) Pf(X)$
- $Pf(\lambda X) = \lambda^n Pf(X)$

$$\begin{tikzcd} E \arrow[d] \\ M \end{tikzcd}$$

real, oriented v.bun., rk = 2n. Let (,) be a fiber metric

Thm (generalized Chern-Gauss-Bonnet thm / special case of Mathai-Quillen)

If $\mathcal{A}$ is a connection in E compatible with (,), then the Euler class of E is

$$\boxed{e^{\mathbb{R}}(E) = \left[\frac{1}{(2\pi)^n} \mathrm{Pf}(F_{\mathcal{A}})\right]} \in H^{2n}_{\text{de Rham}}(M;\mathbb{R})$$

⌈**Proof** (up to sign)

$$e(E)^2 \underset{\substack{\uparrow\\ \text{we know this}}}{=} P_n(E) = (-1)^n \cdot \left(\frac{-1}{2\pi i}\right)^{\frac{4n}{2}} \Psi\big(\overset{\overset{\text{Inv}(O(2n))}{\Downarrow}}{\underset{X \mapsto \det X}{\sigma_{2n}}}, E\big) = \frac{1}{(2\pi)^{2n}}\left[\det F_{\mathcal{A}}\right]$$

$$= \left[\frac{1}{(2\pi)^n} \mathrm{Pf}\, F_{\mathcal{A}}\right]^2$$

from $P_n(E) = \pm C_{2n}(E_{\mathbb{C}}) = \pm\, e(\underbrace{(E_{\mathbb{C}})_{\mathbb{R}}}_{E \oplus E}) = e(E)^2$

⌋

Cor: for M oriented $\overset{\text{Riemannian}}{\vee}$ cpt 2n-mfd, $E = TM$

$$\frac{1}{(2\pi)^n}\int_M \mathrm{Pf}(\underset{\substack{\uparrow\\ \text{any metric connection}\\ \text{(e.g. Levi-Civita)}}}{F_{\mathcal{A}}}) = \underbrace{\chi(M)}_{\substack{\text{Euler}\\ \text{characteristic}}} \in \mathbb{Z}$$

← Chern-Gauss-Bonnet theorem

- case n = 1 : M a closed surface with Riem. metric,

$$\frac{1}{2\pi}\int_M \underset{\substack{\uparrow\\ \text{scalar}\\ \text{curvature}}}{K}\, \underset{\substack{\uparrow\\ \text{metric}\\ \text{area form}}}{d\mathrm{vol}} = \chi(M)$$

- Gauss-Bonnet theorem.

$$\| $$

$$\frac{1}{2\pi}\int_M \underbrace{(F_{\mathcal{A}})}_{\in \Omega^2(M, so(2))}{}_{12} \qquad \begin{pmatrix} 0 & \cdot \\ -\cdot & 0 \end{pmatrix}$$

E.g. for n = 1:

$p = \mathrm{Pf} : \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \longmapsto a$ is not inv. under conjugation by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Rem $X \longmapsto \mathrm{Pf}(X)$ is an invar. polynomial on $\underline{\underline{S}}O(2n)$ but not on $O(2n)$

$\leftrightsquigarrow$ Euler class is defined for <u>oriented</u> bundles.

- Summary of examples of $\mathrm{Inv}(\mathfrak{g})$ $\left(\overset{\text{for } \mathfrak{g} \text{ compact}}{\simeq} H^\bullet(BG) \right)$

  - $\mathrm{Inv}(\mathfrak{u}(n)) = \{\text{sym. polynomials in eigenvalues } \lambda_1, \dots, \lambda_n\} = \mathbb{R}[\sigma_1, \dots, \sigma_n] = \mathbb{R}[s_1, \dots, s_n]$

    $\mathrm{Inv}(\mathfrak{gl}(n,\mathbb{C}))$ - similar

    $\sigma_i = (-1)^i \left(i^{-\text{th}} \text{ coeff of char. poly for } X \right)$

    $\sigma_1 = \mathrm{tr}\, X, \quad \sigma_n = \det X$

    $S_i = \mathrm{tr}\, X^i$

  - $\mathrm{Inv}(\mathfrak{su}(n)) = \mathbb{R}[\underline{\sigma_2}, \dots, \sigma_n]$

    $\mathrm{Inv}(\mathfrak{sl}(n,\mathbb{C}))$ - similar

  - $\mathrm{Inv}(\mathfrak{o}(n)) = \mathbb{R}[\sigma_2, \sigma_4, \dots, \sigma_n] \leftarrow \{\text{sym. poly. on a } \overset{\text{e/v}}{\text{spectrum}} \text{ with } \{\lambda_i\} = \{-\lambda_i\}\}$

    $= \mathbb{R}[S_2, S_4, \dots, S_n]$

  - $\mathrm{Inv}(\mathfrak{so}(2n)) = \mathbb{R}[\sigma_2, \sigma_4, \dots, \sigma_{2n}; \underset{\substack{\uparrow \\ \deg = n}}{\varepsilon}] / \varepsilon^2 = \sigma_{2n}$ $\quad \varepsilon = \mathrm{Pf}\, X$

  - $\mathrm{Inv}(\mathfrak{so}(2n+1)) = \mathbb{R}[\sigma_2, \sigma_4, \dots, \sigma_{2n}]$

Rem/Warning

$$\begin{array}{ccc} \mathrm{Inv}(\mathfrak{gl}(2,\mathbb{R})) = \mathbb{R}[\sigma_1, \sigma_2] & \not\cong & \mathrm{Inv}(O(2)) = \mathbb{R}[\sigma_2] \\ \not\cong\, ! & & \cong \\ H^\bullet(BGL(2,\mathbb{R}),\mathbb{R}) & \cong & H^\bullet(BO(2),\mathbb{R}) \end{array}$$

can fail to be iso. for G non-compact!