Plan
- Hamiltonian & symplectic v. fields, Poisson bracket
- integrable systems, Lagrangian foliations, (Lagr. tori), Arnold-Liouville thm
- examples (Kepler system ...)
- Ham. group actions.

LAST TIME

For (M, ω) symp mfd, $H \in C^\infty(M)$,

$X \in \mathfrak{X}(M)$ is a Hamiltonian v. field with Hamiltonian H if

$\iota_X \omega = dH$

Ex: For N a mfd (not symplectic), $v \in \mathfrak{X}(N) = \Gamma(N, TN)$ a vector field

$\hat{v} \in C^\infty(T^*N)$ - function linear in fibers given by $\underset{\underset{N}{\cap}}{\hat v}(\underset{}{x}, \underset{\underset{T^*_x N}{\cap}}{\xi}) := \langle v(x), \xi \rangle$

Then the hamiltonian v.f.

$$\tilde{v} = X_{\hat{v}} \in \mathfrak{X}(T^*N) \quad \text{is a } \underline{\text{cotangent lift}} \text{ of } v \text{ from } N \text{ to } T^*N.$$

$$(\mathrm{Flow}_t(\tilde v) = \text{cotangent lift of } \mathrm{Flow}_t(v))$$

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def a vector field X on a symp. mfd. (M,ω) is said to be a <u>symplectic</u> v. field if $\boxed{\mathcal{L}_X \omega = 0}$.

(This again implies that $\mathrm{Flow}_t(X)$ is a family of symplectomorphisms)

- a Ham. v. field X_H is symplectic (but a symp. v.f. is not necessarily Hamiltonian)

$$0 \to \mathfrak{X}_{Ham}(M,\omega) \hookrightarrow \mathfrak{X}_{Symp}(M,\omega) \twoheadrightarrow H^1_{dR}(M) \to 0$$
$$X \longmapsto [\iota_X \omega]$$

- Short exact sequence of <u>Lie algebras</u>.

<u>Lie algebra structure</u> on $\mathfrak{X}_{Symp}$ is the usual Lie bracket of vector fields

$(X, Y \in \mathfrak{X}_{Symp} \Rightarrow \mathcal{L}_{[X,Y]}\omega = [\mathcal{L}_X, \mathcal{L}_Y]\omega = 0 \Rightarrow [X,Y] \in \mathfrak{X}_{Symp})$

<u>moreover</u>: $\mathcal{L}_{[X,Y]}\omega = [\mathcal{L}_X, \iota_Y]\omega = d\,\iota_X\iota_Y\omega + \iota_X \underbrace{d\,\iota_Y\omega}_{\mathcal{L}_Y\omega = 0} = d(\iota_X\iota_Y\omega)$

$\Rightarrow$ for X, Y symplectic, $[X,Y]$ is <u>Hamiltonian</u> with $H = \iota_X\iota_Y\omega = \omega(Y,X)$.

<u>Specializing to Hamiltonian v.fields</u>:

For $f, g \in C^\infty(M)$ (symp mfd), define $\{f,g\} := -\omega(X_f, X_g) = X_f(g)$ - "Poisson bracket" of f,g.

opposite convention to Cannas da Silva!

Then: $[X_f, X_g] = X_{\{f,g\}}$.

Lemma $C^\infty(M)$, $\{-,-\}$ is a Lie algebra (over $\mathbb{R}$), i.e.

- $\{-,-\}$ is $\mathbb{R}$-linear in both slots.
- $\{f,g\} = -\{g,f\}$
- $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$ (Jacobi identity)

def A <u>Poisson algebra</u> P
is a commutative assoc. algebra P with
a Lie bracket $\{-,-\}$ satisfying $\{f,gh\} = \{f,g\}h + g\{f,h\}$ (Leibnitz rule)

- For (M,ω) symplectic, $C^\infty(M)$ is a Poisson algebra
- $0 \to \mathbb{R} \hookrightarrow C^\infty(M) \twoheadrightarrow \mathfrak{X}_{ham}(M) \to 0$ SES of Lie algebras.
  0 $\{\cdot,\cdot\}$ $[\cdot,\cdot]_{Lie}$

<u>Integrable systems</u>

def A <u>Hamiltonian system</u> is a triple (M,ω,H) consisting of a symp. mfd (M,ω)
and a Hamiltonian function $H \in C^\infty(M)$

- a function f s.t. $\{f,H\} = 0$ is called an <u>integral of motion</u>

(f is an int. of motion $\Leftrightarrow$ f is constant along integral curves of X_H – see last exercise sheet)

Functions $f_1, \ldots, f_k$ on M are said to be "independent"
if $(df_1)_p, \ldots, (df_k)_p$ are lin. indep. almost everywhere on M (in an open dense subset)

def A Hamiltonian system (M,ω,H) is <u>integrable</u> if it possesses $n = \frac{1}{2}\dim M$
independent integrals of motion, $f_1 = H, f_2, \ldots, f_n$ satisfying
$\{f_i, f_j\} = 0$ for all i,j.
"integrals of motion <u>in involution</u>"

Ex: if $\dim M = 2$ and $dH \neq 0$ almost everywhere, then (M, ω, H) is an integrable system.

E.g. $(\underset{q,p}{\mathbb{R}^2} = T^*\mathbb{R},\ \omega = dq\wedge dp,\ H = \frac{q^2+p^2}{2})$ – harmonic oscillator

Ex: $(M = T^*(\underset{p\quad q}{\mathbb{R}^2\setminus\{0\}}),\ \omega = \omega_{can},\ \overset{f_1:=}{H = \frac{p^2}{2} - \frac{1}{\|q\|}},\ \overset{f_2:=}{J = \underset{q_1p_2 - p_1q_2}{(q\times p)_z}})$

– Kepler system is an int. system

Lemma Let (M, ω, H) be an int. sys. with integrals of motion $f_1 = H, f_2, \dots, f_n$.
Let $c \in \mathbb{R}^n$ be a reg. value of $f = (f_1, \dots, f_n) : M \to \mathbb{R}^n$.
Then a) $f^{-1}(c) \subset M$ is a Lagrangian submanifold ← Exercise from last Friday

b) If Ham. v.fields $X_{f_1}, \dots, X_{f_n}$ are complete on $f^{-1}(c)$, then the connected components of $f^{-1}(c)$ are homog. spaces for $\mathbb{R}^n$, i.e. are of form, (diffeo to) $\mathbb{R}^{n-k}\times\mathbb{T}^k$ for some $0 \le k \le n$, where $\mathbb{T}^k = \underbrace{S^1\times\dots\times S^1}_{k}$ is the k-torus

(follow X_{f_i} to obtain coords)

Thus, any compact component of $f^{-1}(c)$ is a torus $\mathbb{T}^n$!
These components are called Liouville tori.
(E.g. if one of f_i's is proper, then all $f^{-1}(c)$ are compact)

Theorem (Arnold-Liouville) Let (M, ω, H) be an int. sys. with integrals of motion $f_1 = H, f_2, \dots, f_n$. Let $c \in \mathbb{R}^n$ be a reg. value of $f = (f_1, \dots, f_n) : M \to \mathbb{R}^n$. The corresponding level set $f^{-1}(c)$ is a Lagr. submanifold of M.

(a) If the flows of $X_{f_1}, \dots, X_{f_n}$ starting at a point $p \in f^{-1}(c)$ are complete, then the conn. component of $f^{-1}(c)$ containing p is a homogeneous space for $\mathbb{R}^n$. With respect to this affine structure, that component has coords $\varphi_1, \dots, \varphi_n$ – "angle coords", in which the flows of vect. fields $X_{f_1}, \dots, X_{f_n}$ are linear.

(b) There are coords $\psi_1, \dots, \psi_n$ - "<u>action</u> coords" complementary to angle coords, s.t. ψ_i's are integrals of motion and $\varphi_1, \dots, \varphi_n, \psi_1, \dots, \psi_n$ form a Darboux chart

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Thus, dynamics of an int. sys. is very simple and the system has an explicit solution in action-angle coordinates.

- in a neigborhood of a reg. value $c \in \mathbb{R}^n$, $f: M \to \mathbb{R}^n$ is a <u>Lagrangian fibration</u> (i.e. is locally trivial and fibers a Lagr. submanifolds)

a) $\Rightarrow \exists$ coords φ_i on fibers in which flows of X_{f_i} are linear

b) $\Rightarrow \exists$ coord ψ_i on $\mathbb{R}^n$ s.t. $\{\psi_i, f_j\} = 0$, $\{\varphi_i, \psi_j\} = \delta_{ij}$

Note: $\varphi_1, \dots, \varphi_n, f_1, \dots, f_n$ is (generally) not a Darboux chart

Hamiltonian group actions and moment maps

[§.22 in Cannas da Silva]

Let (M,ω) be a symp. mfd, G a Lie group with Lie algebra $\mathfrak{g}$,

$\Psi: G\times M \to M$ a smooth action of G on M s.t. $\Psi_g: M\to M$ is a symplecto-morphism $\forall g\in G$

Let $\psi: \mathfrak{g} \to \mathfrak{X}(M)$ be the corresponding infinitesimal action,

i.e. $\psi_p(\underset{\underset{\mathfrak{g}}{\in}}{\xi}) = \frac{d}{dt}\Big|_{t=0} \Psi(\underbrace{e^{t\xi}}_{\in G}, p)$ or: $\psi_p(\xi) = (d\Psi)_{1,p}(\xi, 0)$

$T_pM \leftarrow \mathfrak{g}\times T_pM$

Case $G=\mathbb{R}$ we have a bijection

$$\{\text{symp. actions of } \mathbb{R} \text{ on } M\} \underset{1-1}{\longleftrightarrow} \{\text{complete symplectic vect. fields on } M\}$$

$$\Psi \longmapsto X_p = \frac{d\Psi_t(p)}{dt}\Big|_{t=0}$$

$$\Psi = Flow_t(X) \longleftarrow X$$

The action Ψ is "Hamiltonian" if the corresp. v.f. X is Hamiltonian, i.e. $\iota_X\omega = dH$ for some $H: M\to \mathbb{R}$.

Case $G=S^1$

S^1-action on M $=$ $\mathbb{R}$-action Ψ on M with $\Psi_{2\pi} = \Psi_0$.

S^1 action is called Hamiltonian if the corresp. $\mathbb{R}$-action is Hamiltonian.