Hamiltonian group actions and moment maps

[S. 22 in Cannas da Silva]

Let (M,ω) be a symp. mfd, G a Lie group with Lie algebra $\mathfrak{g}$,
$\Psi : G \times M \to M$ a smooth action of G on M s.t. $\Psi_g : M \to M$ is a symplecto-morphism $\forall g \in G$

Let $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ be the corresponding infinitesimal action,
i.e. $\underset{\substack{\cap \\ \mathfrak{g}}}{\psi_p(\xi)} = \frac{d}{dt}\Big|_{t=0} \Psi(\underbrace{e^{t\xi}}_{\in G}, p)$ or: $\psi_p(\xi) = (d\Psi)_{1,p}(\xi, 0)$

$T_pM \leftarrow \mathfrak{g} \times T_pM$

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General case: The action Ψ of G on M is a Hamiltonian action if there exists a map $\mu : M \to \underbrace{\mathfrak{g}^*}_{\text{dual of the Lie alg. of } G}$ satisfying:

(a) $\forall \xi \in \mathfrak{g}$,
$\boxed{L_{\psi(\xi)}\omega = d\underbrace{\langle \mu, \xi \rangle}_{\in C^\infty(M)}}$ i.e. the infinitesimal action by ξ is a Ham. v.f. with Hamiltonian $\langle \mu, \xi \rangle$

Then (M, ω, G, μ) is called a Hamiltonian G-space with moment map μ.

Moment map μ is said to be G-equivariant, if

(b) $\boxed{\mu \circ \Phi_g = Ad^*_g \circ \mu} \quad \forall g \in G$

One also has the "comoment map" $\mu^* : \mathfrak{g} \longrightarrow C^\infty(M)$

$\xi \longmapsto \langle \mu, \xi \rangle$

(a) $\mu^*(\xi)$ = Hamiltonian for the v.f. $\psi(\xi)$

equivariance

(b) $\mu^* : \underset{[\,,\,]}{\mathfrak{g}} \longrightarrow \underset{\{\,,\,\}}{C^\infty(M)}$ is a Lie algebra homomorphism:

Poisson bracket

$\mu^*([\xi,\eta]) = \{\mu^*(\xi), \mu^*(\eta)\}$

$\overset{\shortparallel}{\langle \mu, [\xi,\eta] \rangle}$ $\qquad \overset{\shortparallel}{X_{\langle \mu,\xi\rangle} \langle \mu,\eta\rangle}$

$(\psi(\xi)) \overset{\shortparallel}{)} \langle \mu, \eta \rangle$

$\overset{\shortparallel}{\frac{d}{dt}\Big|_{t=0} \Psi^*_{e^{t\xi}} \langle \mu,\eta \rangle}$

$\overset{\shortparallel}{\langle ad^*_\xi \circ \mu, \eta \rangle}$ ← equivariance of μ

Ex: $G = S^1$ (or $\mathbb{R}$)

$\mathfrak{g} = \mathbb{R}$, $\mathfrak{g}^* = \mathbb{R}$, $\mu : M \to \mathbb{R}$

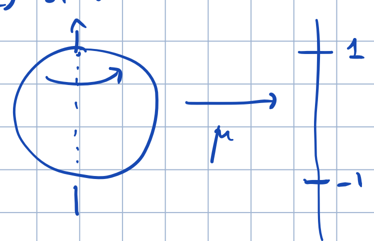
for $\xi = 1 \in \mathfrak{g}$, (a): $\langle \mu, \xi \rangle = \mu \overset{=H}{} $ - Hamiltonian for the vector field $X = \psi(1)$

(b) $\mu \circ \Psi_g = \mu$ - Hamiltonian is constant along int. curves of X

Ex. of Ex: $S^1 \circlearrowright S^2 \xrightarrow{\mu = z} \mathbb{R}$ ← height function

rotations about z axis

$\omega = d\theta \wedge dz \quad X = \frac{\partial}{\partial \theta}$



Ex: $G = \mathbb{T}^n$ - n-torus

$\mathfrak{g} = \mathbb{R}^n$, $\mathfrak{g}^* = \mathbb{R}^n$

$\mu = (\mu_1, \dots, \mu_n)$ (a) μ_i = Ham. for the v.f. $\psi(e_i)$

basis vector in $\mathbb{R}^n$

(b) μ is invariant under $\mathbb{T}^n$-action. ← does not follow from (a)!

Ex: $M = \underset{q}{S^1} \times \underset{p}{S^1}$, $\omega = dq \wedge dp$ $\quad G = S^1 \times S^1$, $\mu_1 = q$, $\mu_2 = p$

$X_{\mu_1} = -\partial_p \quad X_{\mu_2} = \partial_q$ - Ham. v.f. commute but $\{\mu_1, \mu_2\} \neq 0$

Ex: translations/momentum

$\underset{G}{\overset{\shortparallel}{\mathbb{R}^3}} \overset{\Phi}{\circlearrowright} \underbrace{T^*\mathbb{R}^3}_{M}$, $\omega = \sum_i dq_i \wedge dp_i$, $\quad \Phi_{\vec{a}}(\vec{q}, \vec{p}) = (\vec{q} + \vec{a}, \vec{p})$

$\vec{a} \in G$

$\underset{\mathfrak{g}}{\psi(\vec{a})} = (\vec{a}, 0) = X_{\langle \vec{a}, \vec{p} \rangle}$

$\overset{\shortparallel}{\sum a_i \frac{\partial}{\partial q_i}}$

$\Rightarrow$ moment map is

$\mu : M \longrightarrow \mathfrak{g}^*$

$(\vec{q}, \vec{p}) \longmapsto \vec{p}$

- it is G-invariant.

Ex: $\underset{\underset{G}{\shortparallel}}{SO(3)} \overset{\Phi}{\circlearrowright} \underbrace{T^*\mathbb{R}^3}_{M}$, $\Phi_{R_{\vec{n},\theta} \in SO(3)}(\vec{q},\vec{p}) = (R_{\vec{n},\theta}\,\vec{q}\,,\, R_{\vec{n},\theta}(\vec{p}))$

rotation about $\vec{n} \in S^2 \subset \mathbb{R}^3$
by angle θ

infinitesimal action:

$$\psi(\vec{\xi}) = (\vec{\xi}\times\vec{q}\,,\,\vec{\xi}\times\vec{p}) = X_{\langle\vec{\xi},\,\vec{q}\times\vec{p}\rangle}$$

$\vec{\xi} = \theta\vec{n} \in so(3)$

$\Rightarrow$
$$\mu : T^*\mathbb{R}^3 \longrightarrow so(3)^* \simeq \mathbb{R}^3$$
$$(\vec{q},\vec{p}) \longmapsto \vec{q}\times\vec{p}$$

- moment map
(satisfies equivariance - simple check)

Rem Given a Ham. group action $G \circlearrowright (M,\omega)$ with moment map μ, (not necessarily equivariant)
we can change $\mu \longrightarrow \mu' = \mu + v$

v - loc. constant function on M with values in $\mathfrak{g}^*$.

(preserves the Hamiltonian property (a) but can equivariance (b) if it were satisfied)

Ex: coadjoint orbits

$G \underset{Ad^*}{\circlearrowright} \mathfrak{g}^*$ - coad. action, $\langle Ad^*_g(\xi), X\rangle = \langle \xi, Ad_{g^{-1}}(X)\rangle$, $g \in G$, $\xi \in \mathfrak{g}^*$, $X \in \mathfrak{g}$

coad. infinitesimal action: $\mathfrak{g} \underset{ad^*}{\circlearrowright} \mathfrak{g}^*$, $\langle ad^*_X(\xi), Y\rangle = -\langle \xi, [X,Y]\rangle$

Let $\mathcal{O} \subset \mathfrak{g}^*$ be an orbit of coad. action of G
$\mathcal{O} = \{Ad^*_g(\xi) \mid g \in G\}$, ξ - fixed.

We have $0 \to \text{stab}(\xi) \hookrightarrow \mathfrak{g} \xrightarrow{\sigma} T_\xi\mathcal{O} \to 0$.

$\text{stab}(\xi) = \{X \in \mathfrak{g} \mid ad^*_X\xi = 0\}$, $X \longmapsto ad^*_X$, $T_\xi\mathcal{O} \subset T_\xi\mathfrak{g}^* \simeq \mathfrak{g}^*$

skew-sym bilinear form $\tilde{\omega}_\xi : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$
$(X,Y) \longmapsto \langle \xi, [X,Y]\rangle$

has $\text{stab}(\xi)$ as kernel
$\rightsquigarrow$ induces a nondeg. skew-sym bilinear form ω_ξ on $T_\xi\mathcal{O}$.

in fact: $\{\omega_\xi\}_{\xi\in\mathcal{O}}$ defines
a symplectic structure ω on $\mathcal{O}$
(Kirillov - Kostant - Souriau)

In fact, the action $G \circlearrowright \mathcal{O}$ is Hamiltonian, with
Ad^*

equivar. moment map $\mu: \mathcal{O} \longrightarrow \mathfrak{g}^*$ the tautological inclusion of the orbit into $\mathfrak{g}^*$.
$\mu = i$

(a) infin. action $\psi: \mathfrak{g} \to \mathfrak{X}(\mathcal{O})$

$$X \longmapsto (\xi \mapsto \mathrm{ad}^*_X(\xi))$$

$$\iota_{\psi(X)}\omega \overset{\text{WANT}}{=} d\langle \mu, X\rangle$$

$\Big\}$ evaluate at $\xi \in \mathcal{O}$ and pair with a tangent vector $\mathrm{ad}^*_Y(\xi)$

(b) equivariance of the inclusion - trivial

$$\langle \xi, [X,Y]\rangle = \langle \xi, [X,Y]\rangle \quad \checkmark$$

<u>Example</u> (Atiyah-Bott) Let Σ - closed oriented surface

G a Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$ with ad-invariant (non-deg.) inner product $(\,,\,)$.

Then: $\mathrm{Conn} = \left\{ \text{connections } A \text{ in } \begin{matrix}\Sigma\times G\\ \downarrow \text{ triv.}\\ \Sigma\end{matrix} \right\}$ form a (co-dim.) symplectic manifold,

$$T_A \mathrm{Conn} \simeq \Omega^1(M,\mathfrak{g})$$

$$\omega_A(\underbrace{\alpha,\beta}_{\Omega^1(M,\mathfrak{g})}) := \int_\Sigma (\alpha \overset{\wedge}{;} \beta)$$

$\mathrm{Map}(\Sigma,G) \circlearrowright \mathrm{Conn}$ - action of bundle automorphisms on connections
$= \mathcal{G}$

$$\left(\underset{\sigma^* A}{A} \to {}^g A = g A g^{-1} + g\, dg^{-1} = \mathrm{Ad}_g A + g^*\theta_{\text{Maurer-Cartan}} \right)$$

for a matrix group G

- This action is <u>Hamiltonian</u>,

with moment map $\mu: \mathrm{Conn} \xrightarrow{\text{curvature}} \Omega^2(M,\mathfrak{g})$

$$A \longmapsto F_A = dA + \tfrac{1}{2}[A,A]$$

<u>Idea:</u> $\underset{\mathrm{Map}(\Sigma,\mathfrak{g}) = \mathrm{Lie}(\mathcal{G})}{\psi(\xi)}\Big|_{A\in\mathrm{Conn}} = -d\xi - [A,\xi] \in T_A\mathrm{Conn} \simeq \Omega^1(M,\mathfrak{g})$ - vector field of infin. action, evaluated at a point $A \in \mathrm{Conn}$

$$\underset{\text{any}}{\iota_\rho}\, \iota_{\psi(\xi)}\omega\Big|_A = -\int_\Sigma (d\xi + [A,\xi], \underset{\Omega^1(M,\mathfrak{g}) = T_A\mathrm{Conn}}{\rho}) \quad \checkmark$$

$$\iota_\rho \delta \langle \mu, \xi\rangle = \int_\Sigma \iota_\rho\delta(\xi, dA + \tfrac{1}{2}[A,A]) = \int_\Sigma (\xi, d\rho + [\rho, A]) \underset{\text{Stokes'}}{=} -\int_\Sigma (d\xi + [A,\xi], \rho)$$

$\Rightarrow \langle \mu, \xi\rangle$ is a Hamiltonian for $\psi(\xi)$

- μ is equivariant (by transformation property of curvature under gauge transformation) 5