LAST TIME

Delzant theorem

$\{\text{toric } 2n\text{-manifolds}\} \xleftrightarrow[1-1]{} \{\text{Delzant polytopes in } \mathbb{R}^n\}$

$(M^{2n}, \omega, \mathbb{T}^n, \mu) \longmapsto \Delta = \mu(M)$

Construction $\Delta \longrightarrow M_\Delta$

polytope in $\mathbb{R}^n$ w/ d faces

$\Delta = \{x \in (\mathbb{R}^n)^* \mid \langle x, v_i \rangle \geq \lambda_i \}$

$v_i \in \mathbb{Z}^n$ face vectors

Set $\pi : \mathbb{R}^d \to \mathbb{R}^n$

$e_i \longmapsto v_i$

std basis

Consider LES

$\underset{\ker \pi}{H} \xhookrightarrow{i} \mathbb{T}^d \xrightarrow[\pi]{} \mathbb{T}^n$ groups $\quad \sim \quad \mathfrak{h} \xhookrightarrow{i} \mathbb{R}^d \xrightarrow{\pi} \mathbb{R}^n \longrightarrow (\mathbb{R}^n)^* \xhookrightarrow{\pi^*} (\mathbb{R}^d)^* \xrightarrow{i^*} \mathfrak{h}^*$

LES of ab. Lie alg

Take $\quad \mathbb{T}^d \underset{\text{std action}}{\circlearrowright} (\mathbb{C}^d, \omega_0) \xrightarrow{\varphi} (\mathbb{R}^d)^*$

std symp form

$(z_1, \dots, z_d) \longmapsto (|z_1|^2 + \lambda_1, \dots, |z_d|^2 + \lambda_d)$

For the sub-torus $H \subset \mathbb{T}^d$, this induces

$\underset{\dim = (d-n)}{H} \circlearrowright (\mathbb{C}^d, \omega_0) \xrightarrow[i^* \circ \varphi]{} \underset{\dim = (d-n)}{\mathfrak{h}^*}$

Construct $\quad M_\Delta := \mathbb{C}^d /\!/ H = \mu^{-1}(0) / H$

comes with eff., ham. action of $\mathbb{T}^d / H = \mathbb{T}^n$, with moment map $\mu : M_\Delta \longrightarrow \ker \pi^* = (\mathbb{R}^n)^*$

induced from φ.

claim: $\mu(M_\Delta) = \Delta$.

Ex: $\Delta = [0,a] \subset \mathbb{R}^*$ $\quad v_\phi = v^{1,1}$ ← standard basis vector in $\mathbb{R}$ $\quad v_2 = -v$

$n=1,\ d=2$ $\quad$ 0 $\quad$ a

$\Delta:\ \langle x, v_1\rangle \geq 0 \qquad v_1 = v$

$\qquad \langle x, v_2\rangle \geq -a \qquad v_2 = -v$

$T^2 \circlearrowright \mathbb{C}^2 \xrightarrow{\varphi} (\mathbb{R}^2)^*$

$(z_1, z_2) \mapsto (|z_0|^2, |z_1|^2 - a)$

projection

$\pi: \mathbb{R}^2 \twoheadrightarrow \mathbb{R}$ $\qquad \ker \pi = \mathrm{Span}(e_1 + e_2)$ $\qquad$ so, H = diagonal subgroup $\{(\alpha,\alpha)\}$ of T^2

$e_1 \longmapsto -v$

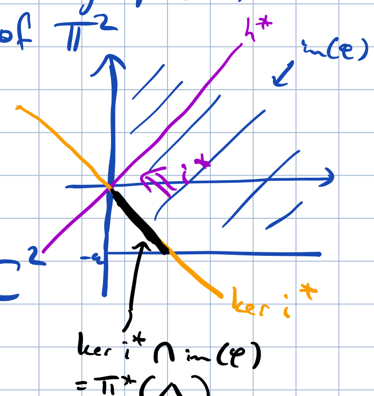
$e_2 \longmapsto v$

LES: $0 \to H \xhookrightarrow{i} T^2 \xrightarrow{\pi} S^1 \to 0$

$0 \leftarrow h^* \xleftarrow{i^*} (\mathbb{R}^2)^* \xleftarrow{\pi^*} \mathbb{R}^* \leftarrow 0$

$x_1 + x_2 \longleftarrow (x_1, x_2)$

$\qquad (y,-y) \longleftarrow y$



Action of diagonal subgroup $H = \{(e^{2\pi i\theta}, e^{2\pi i\theta}) \in S^1 \times S^1\}$ on $\mathbb{C}^2$

$(e^{2\pi i\theta}, e^{2\pi i\theta}) \cdot (z_1, z_2) = (e^{2\pi i\theta} z_1,\ e^{2\pi i\theta} z_2)$

has moment map $\underbrace{(i^* \circ \varphi)}_{\mu_H}(z_1, z_2) = |z_1|^2 + |z_2|^2 - a$

with zero-level set $\quad \mu_H^{-1}(0) = \{(z_1,z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = a\}$

Thus, $M_\Delta = \mu_H^{-1}(0)/H = \mathbb{C}P^1$!

$\mu: M_\Delta \to \mathbb{R}^*$?

$$\begin{array}{ccc} & \mathbb{C}^2 & (z_1,z_2) \\ & \downarrow \varphi & \downarrow \\ h^* \longleftarrow & (\mathbb{R}^2)^* & (|z_1|^2, |z_2|^2 - a) \end{array}$$

$\mathbb{C}P^1 = \{(z_1,z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = a\}/H$

$\downarrow \mu = (\pi^*)^{-1}\varphi$

$\mathbb{R}^* \qquad |z_1|^2$

So, $\mu(\mathbb{C}P^1) = [0,a] = \Delta$.

Rem Another description: use the moment map without λ-shifts

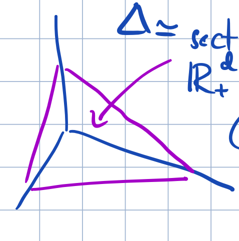
$$\mathbb{C}^d \xrightarrow{\tilde\varphi} (\mathbb{R}^d)^* \xrightarrow{i^*} h^*, \qquad \tilde\mu = i^* \circ \tilde\varphi$$

$(z_1, \dots, z_d) \longmapsto (|z_1|^2, \dots, |z_d|^2)$

Then: $M_\Delta = \tilde\mu^{-1}(-\lambda)/H$

Symp. reduction at nonzero level

$\Delta \simeq$ section of orthant $\mathbb{R}^d_+$ by affine subspace $(i^*)^{-1}(-\lambda)$



ASIDE

Q: Can one always do symp. reduction at nonzero level of moment map?

A: not generally, but:

Lemma

If $\underset{\text{cpt}}{G} \circlearrowright (M,\omega) \xrightarrow{\mu} \mathfrak{g}^*$ Hamiltonian G-space, $\mathcal{O} \subset \mathfrak{g}^*$ a coadjoint orbit

then, assuming G acts freely on $\mu^{-1}(\mathcal{O})$, $\mu^{-1}(\mathcal{O})/G$ is a smooth mfd with a natural symp. str. inherited from ω.

Proof We have $G \circlearrowright (M \times \mathcal{O}, \omega \times \omega_{\mathcal{O}}) \xrightarrow{(x,\xi) \longmapsto \mu(x)+\xi} \mathfrak{g}^*$ – Ham. torus action on $M\times\mathcal{O}$

$g\cdot(x,\xi) = (g\cdot x, Ad_g^*(\xi))$

Kirillov-Kostant symp. str. on $\mathcal{O}$

Then: $$M\times\mathcal{O}/\!\!/G = \frac{\{(x,\xi)\in M\times\mathcal{O} \mid \mu(x)+\xi = 0,\ \xi\in\mathcal{O}\}}{G} \simeq \frac{\{x\in M \mid \mu(x)\in\mathcal{O}\}}{G}$$

symp. mfd by MWM thm.

$\boxtimes$

Corollary: Given $\mathbb{T}^m \circlearrowright (M,\omega) \xrightarrow{\mu} (\mathbb{R}^m)^*$ a Ham. $\mathbb{T}^m$-space, $\lambda \in (\mathbb{R}^m)^*$ any vector,

$\mu^{-1}(\lambda)/\mathbb{T}^m$ is a symp. mfd (provided that action $\mathbb{T}^m \circlearrowright \mu^{-1}(\lambda)$ is free).

$\shortparallel$

$M/\!\!/_\lambda \mathbb{T}^m$

reduction at nonzero level.

• Homology of symplectic toric manifolds

Thm: Let $(M, \omega, \mathbb{T}^n, \mu)$ be a symplectic toric mfd and let $X \in \mathbb{R}^n$ a vector with components independent over $\mathbb{Q}$.

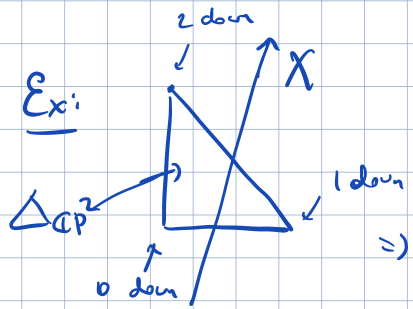
Then: $H_{2k+1}(M, \mathbb{Z}) = 0$,

$H_{2k}(M, \mathbb{Z}) = \mathbb{Z}^{r_k}$, $r_k = \#\{$vertices of Δ which have k adjacent edges pointing down w.r.t. $X\}$

[proof - from - Morse theory: we have a Morse function $\langle X, \mu \rangle$ with crit. points of even index only]

crit. pts of $\langle X, \mu\rangle$ $= \mathbb{T}^n$-fixed pts of M.

Ex:



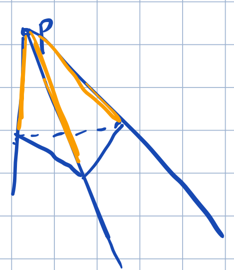
$\Rightarrow$ $H_0 = \mathbb{Z}$, $H_1 = 0$, $H_2 = \mathbb{Z}$, $H_3 = 0$, $H_4 = \mathbb{Z}$ - correct for $\mathbb{CP}^2$

Corollary: Euler characteristic of M $\leftarrow$ symp. toric $= \#\{$vertices of $\Delta\}$

• Symplectic blow-up

Let Δ a Delzant polytope, p - vertex, $u_1, \dots, u_n$ - (primitive) edge vectors at p.

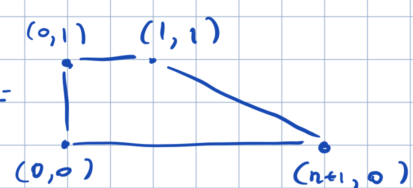
$\Delta_{p,\varepsilon}$:= Δ with corner at p chopped off; vertex p is replaced by n new vertices $p + \varepsilon u_j$, $j = 1 \dots n$, $\varepsilon > 0$



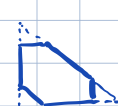
$\Delta_{p,\varepsilon}$ is a new Delzant polytope

corresp. toric manifold is the "ε-symplectic blow-up" of M_Δ.

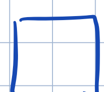
Ex Hirzebruch surface $\mathcal{H}_n$ for $n \geq 0$ = toric mfd with Δ =

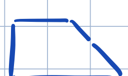


Ex:



- blow-up of $\mathbb{CP}^2$ at 3 points

$\mathcal{H}_0 = M($  $) = \mathbb{CP}^1 \times \mathbb{CP}^1$

$\mathcal{H}_1 = M($  $)$ = symp. blow-up of $\mathbb{CP}^2$

$$\mathcal{H}_n = \left(\mathbb{C}^2 - \{0\}\right) \times \left(\mathbb{C}^2 - \{0\}\right) \Big/ \mathbb{C}^* \times \mathbb{C}^*$$

$(\mathbb{C}^*)^2$-action: $\quad (\lambda, \mu) \cdot (z_0, z_1, w_0, w_1) = (\lambda z_0, \lambda z_1, \mu w_0, \lambda^{-n} \mu w_1)$

- $\mathcal{H}_n$ is a $\mathbb{C}P^1$-bundle over $\mathbb{C}P^1$

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<u>Symplectic blow-up</u> (of $\mathbb{C}^n$, at origin)

Let $\tau = \{([p], z) \mid p \in \mathbb{C}^n - \{0\},\ z = \lambda p$ for some $\lambda \in \mathbb{C}\}$ — taut. bundle over $\mathbb{C}P^{n-1}$

$$\begin{array}{cc} \downarrow & \text{I} \\ \mathbb{C}P^{n-1} \ni & [p] \end{array}$$

Total space τ =: "blow-up of $\mathbb{C}^n$ at origin"

$$\begin{array}{ccc} \tau & & ([p], z) \\ \beta \downarrow & & \downarrow \\ \mathbb{C}^n & & z \end{array}$$

— "blow-down map"

$$\tau = \underbrace{\{([p], 0) \mid p \in \mathbb{C}^n - \{0\}\}}_{E \,\simeq\, \mathbb{C}P^{n-1}} \ \amalg\ \underbrace{\{([p], z) \mid p \in \mathbb{C}^n - \{0\},\ z = \lambda p \text{ for some } \lambda \in \mathbb{C}^*\}}_{S \,\overset{\beta}{\simeq}\, \mathbb{C}^n - \{0\}}$$

"exceptional divisor"

τ = "smooth replacement of 0 in $\mathbb{C}^n$ by $\mathbb{C}P^{n-1}$"

- a "blow-up symplectic form" on τ is a $U(n)$-invariant symp. form ω on τ s.t. $\omega - \beta^* \underbrace{\omega_0}_{\text{std. symp. form on } \mathbb{C}^n}$ is <u>compactly supported</u>.

- two blow-up symp. forms ω_1, ω_2 on τ are "equivalent" iff $\omega_1 = \Phi^* \omega_2$ for $\Phi : \tau \to \tau$ some $U(n)$-equivar. diffeo.

<u>Lemma</u> (Guillemin–Sternberg) two blow-up symp. forms ω_1, ω_2 are equivalent iff $\omega_1|_E = \omega_2|_E$

ε-Symplectic blow-up of $\mathbb{C}^n$ at 0 = (τ, ω)

ω: any blow-up symp. form s.t. $\omega|_E = \varepsilon \cdot \omega_{FS}$

ω_{FS}: Fubini–Study 2-form on $\mathbb{C}P^{n-1}$
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For (M,ω) a symp. mfd, $q \in M$ a point,

one has a chart (by Darboux thm) $(\mathcal{U}, z_1, \dots, z_n)$ s.t. $\omega|_{\mathcal{U}} = \frac{i}{2} \sum_k dz_k \wedge d\bar{z}_k$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \ni q$

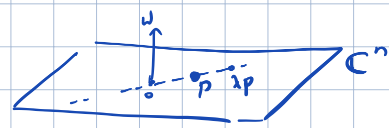
- can perform ε-blow-up of M at q modeled on $\mathbb{C}^n$ at 0,
without changing ω outside of a small nbhd of q.

<u>Example</u> $\mathbb{P}(\tau \oplus \underline{\mathbb{C}})$

$\uparrow$ projectivization

$$\begin{array}{ccc} & \mathbb{P}(\tau \oplus \underline{\mathbb{C}}) & \\ & \swarrow \qquad \searrow^{\beta} & \\ \mathbb{C}P^{n-1} & & \mathbb{C}P^n \end{array}$$

$([p],[\lambda p : w]) \mapsto [\lambda p : w]$ — line in $\mathbb{C}^{n+1}$



- $\beta^{-1}([0:\dots:0:1]) = \underbrace{\{[p],[0:\dots:0:1]\}}_{E \text{ - exceptional divisor}} \simeq \mathbb{C}P^{n-1}$
$\qquad [0:\dots:0:1] \in \mathbb{C}P^n$
- β is a diffeo on the complement $S = \{([p],[\lambda p : w]) \mid [p] \in \mathbb{C}P^{n-1}, \lambda \in \mathbb{C}^*, w \in \mathbb{C}\} \simeq \mathbb{C}P^n \setminus \{[0:\dots:0:1]\}$

Thus: $\mathbb{P}(\tau \oplus \underline{\mathbb{C}})$ is the blow-up of $\mathbb{C}P^n$ at a point $[0:\dots:0:1]$; β - blow-down map.

For $n=2$, this is first Hirzebruch surface $\mathcal{H}_1$.