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- R. Cohen "Bundles, homotopy and manifolds"
- A. Hatcher "Vector bundles and K-theory"

Ref.: · Husemoller "Fibre bundles", 1995
· Milnor-Stasheff "Char. classes", 1975

orig. references
· Steenrod "The topology of fibre bundles", 1951
· Ehresmann "Les connexions infinitésimales dans un espace fibré différentiable", 1951

Bundles and connections

def A fiber bundle is the data
- E - total space
- M - base
- $\pi: E \to B$ projection
- F - standard fiber

where E, M, F are topological spaces and $\pi: E \to M$ is a (continuous) surjective map

Additionally, "local triviality" assumption:

each point $x \in M$ has an open neighborhood $\underset{x}{\underset{\Downarrow}{U}} \subset M$ and a homeomorphism

$$\varphi : U \times F \xrightarrow{\sim} \pi^{-1}(U) \quad \text{s.t.} \quad \pi \circ \varphi = \mathrm{proj}_1 : U \times F \to U$$

"trivializing map"

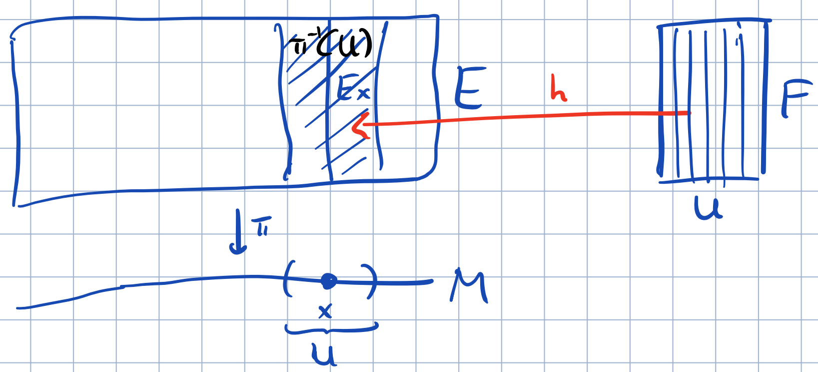
$$\begin{tikzcd} U \times F \arrow[r, "\sim"] \arrow[dr, "\mathrm{proj}_1"'] & \pi^{-1}(U) \arrow[d, "\pi"] \arrow[dl, phantom, "\square"] \\ & U \end{tikzcd}$$

Notation: $\begin{array}{c} E \\ \pi \downarrow \\ M \end{array}$, or $\begin{array}{c} E \leftarrow F \\ \pi \downarrow \\ M \end{array}$

Corollary: Fiber $E_x = \pi^{-1}(x)$ over any point $x \in M$ is homeomorphic to F.

Remark

So, a bundle is "a family of top. spaces $E_x \simeq F$ indexed by $x \in M$"



[· φ gives a parametrization of the fiber by points $\xi \in F$, and a parametrization of E over U by pairs (x, ξ). $\quad x \in U,\ \xi \in F$]

Local trivialization of a fiber bundle: open cover $\{U_\alpha\}$ of M and trivializing maps $\varphi_\alpha : U_\alpha \times F \to \pi^{-1}(U_\alpha)$

Smooth fiber bundle: E, M, F - C^∞ manifolds, π - surjective C^∞ map, maps φ - diffeomorphisms

Exercise: prove that π is automatically a submersion.

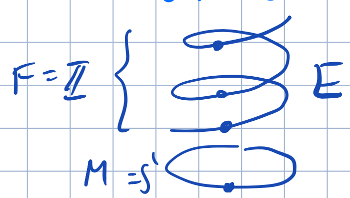
Examples

- **Trivial bundle** over M with fiber F: $\begin{array}{c} E = M \times F \\ \downarrow \pi = \mathrm{proj}_1 \\ B \end{array}$

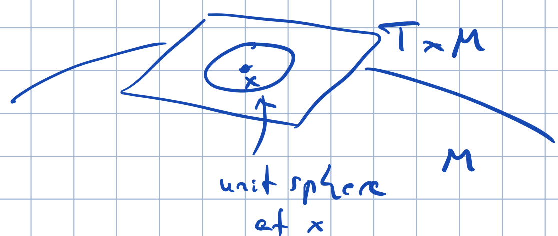
- for an n-manifold M, the tangent bundle $\begin{array}{c} E = TM \\ \downarrow \\ B \end{array}$ is a fiber bundle with fiber $\mathbb{R}^n$.

- any covering space $\pi : E \to M$ over M is a fiber bundle with discrete fiber

  e.g. $\pi : \mathbb{R} \to S^1$ (fiber $F = \mathbb{Z}$)
  $t \mapsto e^{2\pi i t}$

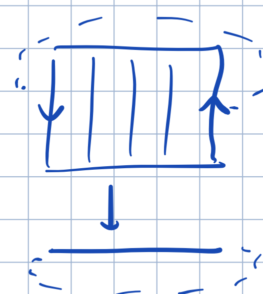


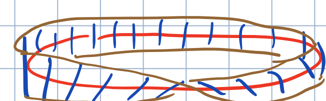
- for M a Riemannian manifold, $STM = \{(x,v) \in TM \mid \langle v,v\rangle = 1\}$
  – "unit tangent (sphere) bundle"



- Möbius strip

$$E = \{(x,y) \in [0,1] \times \mathbb{R}\} \big/ (0,y) \sim (1,-y)$$
$$\downarrow$$
$$M = [0,1] \big/ 0 \sim 1$$





- mapping tori: Σ - mfd, $f : \Sigma \xrightarrow{\sim} \Sigma$ - diffeomorphism

$$T_f = [0,1] \times \Sigma \big/ (1,y) \sim (0, f(y)) \ \forall y \in \Sigma$$

– a bundle over S^1 with fiber Σ.

Projective spaces

unit sphere in $\mathbb{R}^{n+1}$

- $S^n \xrightarrow{\pi} \mathbb{RP}^n = S^n / _{x \sim -x}$

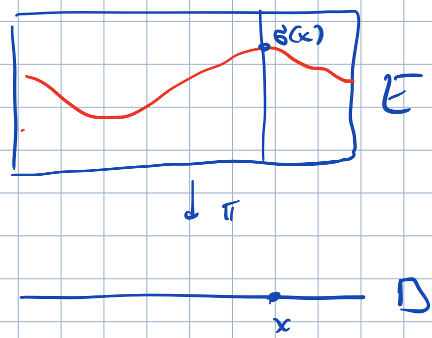
  fiber $= \mathbb{Z}_2$

- $S^{2n+1} = \{(z_0 \ldots z_n) \in \mathbb{C}^{n+1} \mid \sum_0^n |z_i|^2 = 1\} \longrightarrow \mathbb{CP}^n = S^{2n+1} / _{z \sim \lambda z}$, $\lambda \in \mathbb{C}, |\lambda| = 1$

  fiber $= S^1$

def A <u>section</u> of a fiber bundle (E, M, π) is a map $\sigma : B \to E$

s.t. $\pi \circ \sigma = id_M$.



Ex: • a section of TM = a vector field on M

• a section of a trivial bundle $\begin{matrix} M\times F \\ \downarrow \\ M \end{matrix}$ is $\sigma : x \longmapsto (x, f(x))$ for any map $f : M \to F$.

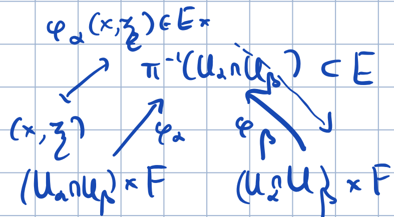
<u>Transition functions</u>

Given a local trivialization $\{(U_\alpha, \varphi_\alpha)\}$ of $\begin{matrix} E \\ \downarrow \pi \\ M \end{matrix}$,

$\varphi_\alpha : U_\alpha \times F \to \pi^{-1}(U_\alpha)$

one has $\varphi_\beta^{-1}\varphi_\alpha : (U_\alpha \cap U_\beta)\times F \longrightarrow (U_\alpha\cap U_\beta)\times F$

$$(x, \xi) \longmapsto (x, t_{\beta\alpha}(x)\,\xi)$$



Here $t_{\beta\alpha} : U_\alpha \cap U_\beta \to G$ - transition function

G - a "<u>structure group</u>" acting on F (effectively (i.e. $\ker(G \to Diff(F)) = \{1\}$)) by diffeomorphisms.

(general case: $G = Diff(F)$, but it could also be a subgroup of $Diff(F)$)

One has: (1) $t_{\alpha\alpha}(x) = 1$

(2) $t_{\alpha\beta}(x) = t_{\beta\alpha}(x)^{-1}$

(3) $t_{\alpha\beta}(x)\, t_{\beta\gamma}(x)\, t_{\gamma\alpha}(x) = 1 \quad \forall\, x \in U_\alpha \cap U_\beta \cap U_\gamma$

(or: $t_{\alpha\beta} t_{\beta\gamma} = t_{\alpha\gamma}$)

"cocycle condition"

- call the data of a cover $\mathcal{U} = \{U_\alpha\}$ and maps $\{t_{\alpha\beta} : U_\alpha \cap U_\beta \to G\}$ satisfying (1),(2),(3) a "(G-)cocycle" on M.

Def: two cocycles $(\{U_\alpha\}_{\alpha\in I}, \{t_{\alpha\beta}\}), (\{U_{\alpha'}\}_{\alpha'\in I'}, \{t_{\alpha'\beta'}\})$ are "equivalent" if there is a common cocycle in which they are both contained.

[the open sets don't need to be distinct]

- cocycles corresponding to a given fiber bundle are equivalent [can change the cover & trivializations]

Vice versa: Given a cover $\{U_\alpha\}$ of M and transition functions satisfying (1), (2), (3), one can construct (glue) a fiber bundle

$$E = \frac{\coprod_\alpha U_\alpha \times F}{\text{for each } \alpha,\beta \text{ and } x \in U_\alpha \cap U_\beta,\ \underset{\underset{U_\alpha\times F}{\cap}}{(x,\xi)_\alpha} \sim \underset{\underset{U_\beta\times F}{\cap}}{(x, t_{\beta\alpha}(x)\xi)_\beta}}$$

def A morphism of fiber bundles $\begin{matrix} E_1 \\ \downarrow \pi_1 \\ M_1 \end{matrix}$, $\begin{matrix} E_2 \\ \downarrow \pi_2 \\ M_2 \end{matrix}$ is a pair of maps $E_1 \xrightarrow{\Phi} E_2$, $M_1 \xrightarrow{f} M_2$

such that the diagram
$$\begin{tikzcd} E_1 \arrow[r, "\Phi"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ M_1 \arrow[r, "f"'] & M_2 \end{tikzcd}$$
commutes.

(in particular, fibers are mapped into fibers: $p, p' \in (E_1)_x \Rightarrow \Phi(p), \Phi(p') \in (E_2)_{f(x)}$)

Rem Fiber bundles $\begin{matrix} E_1 \\ \downarrow \\ M \end{matrix}$, $\begin{matrix} E_2 \\ \downarrow \\ M \end{matrix}$ are isomorphic, with iso. covering the identity map $M \xrightarrow{id} M$

iff local trivializations of E_1, E_2 over the same trivializing cover $\{U_\alpha\}$ are related by

$$(t_2)_{\alpha\beta}(x) = h_\alpha(x)(t_1)_{\alpha\beta}(x)\, h_\beta^{-1}(x) \qquad \forall x \in U_\alpha \cap U_\beta.$$

for some collection of maps $h_\alpha : U_\alpha \to G$

[
$$\begin{tikzcd} \pi_1^{-1}(U_\alpha) \arrow[r, "\Phi"] & \pi_2^{-1}(U_\alpha) \\ U_\alpha \times F \arrow[u, "\varphi^1_\alpha"] \arrow[r, "id\times h_\alpha"] & U_\alpha \times F \arrow[u, "\varphi^2_\alpha"'] \end{tikzcd}$$

$(\varphi^2_\alpha)^{-1}(\varphi^2_\beta) : (x,\xi^2) \mapsto (id\times h_\alpha)\overbrace{(\varphi^1_\alpha)^{-1}\Phi^{-1}\Phi(\varphi_\beta)}^{id\times t^1_{\alpha\beta}}(x, h_\beta^{-1}(\xi^2)) = (x, \underbrace{h_\alpha t^1_{\alpha\beta} h_\beta^{-1}}_{t^2_{\alpha\beta}}(\xi^2))$
]

In particular, $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ is iso. to the trivial bundle iff $t_{\alpha\beta}(x) = h_\alpha(x)\, h_\beta^{-1}(x)$ for some functions $\{h_\alpha : U_\alpha \to G\}$

Vector bundles

def A vector bundle is a fiber bundle $\begin{array}{c} E \\ \downarrow \pi \\ M \end{array}$ where $F = V$ is a vector space, (over $k = \mathbb{R}$ or $\mathbb{C}$)

and transition functions $t_{\alpha\beta}$ - (invertible) linear transformations of V (i.e. $G = GL(V)$).

<put another way, F is linear, fibers E_x are linear and diffeos φ induce linear isos $F \to E_x$>

- $\dim V$ = "rank" of the vector bundle.
- rank 1 vector bundles are also called "line bundles".
- rank k vector bundle = "k-plane bundle"

Ex: • tangent bundle TM of a smooth n-mfd M

– rank n vector bundle over $\mathbb{R}$

if $\{\underbrace{(U_\alpha, \psi_\alpha : M \to \mathbb{R}^n)}_{\text{coord charts}}\}$ an atlas, loc. trivialization:

$$\{(U_\alpha, \varphi_\alpha : U_\alpha \times \mathbb{R}^n \to TM)\}$$
$$(x, \{v^i\}) \mapsto (x, \sum_i v^i \underbrace{\frac{\partial}{\partial x^i_{(\alpha)}}}_{\text{coord. basis in } T_xM})$$

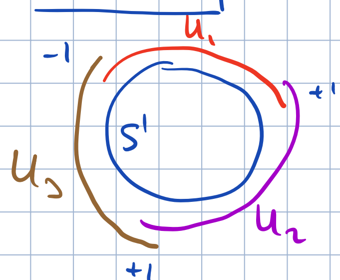
transition functions:

$$v^i_{(\alpha)} = \sum_j \frac{\partial x^i_{(\alpha)}}{\partial x^j_{(\beta)}} v^j_{(\beta)}$$

So: $t_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n)$

$$x \longmapsto \left(\frac{\partial x^i_{(\alpha)}}{\partial x^j_{(\beta)}}\right)_{ij}$$ - Jacobian matrix

• Möbius strip: rank 1 vector bundle over S^1 defined by transition functions



$\leftarrow$ trivializing cover

with transition functions

$$t_{12} = +1, \quad t_{23} = +1, \quad t_{31} = -1$$

Exercise: (a) prove that it is not iso to the trivial bundle

(b) consider the product bundles $E_1 = MS \times (S^1 \times \mathbb{R})$, $E_2 = (S^1 \times \mathbb{R}) \times MS$

Möbius strip

$$\begin{array}{ccc} E_1 & & E_2 \\ \downarrow & & \downarrow \\ S^1 \times S^1 & & S^1 \times S^1 \end{array}$$

show that there is no iso (covering identity on the base) between E_1, E_2, but there is an invertible morphism $E_1 \to E_2$ which does not cover identity.