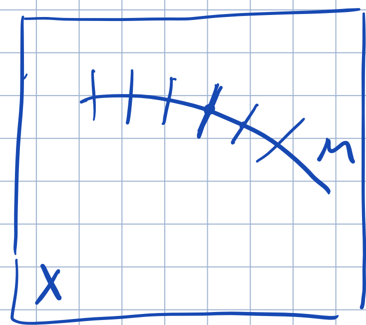
• **Normal bundle** Let $i: M \xrightarrow[\text{immersion}]{} X$ ← Riemannian mfd

then the normal bundle is: $\boxed{NM = \bigcup_{x\in M} \underbrace{(i_* T_x M)^\perp}_{N_x M}}$ ortlog. complement in $T_{i(x)} X$

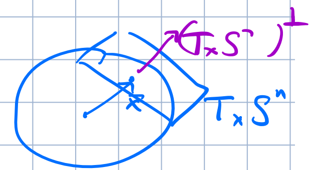
$$\begin{array}{c} NM \\ \downarrow \\ M \end{array}$$



$\mathrm{rank}(NM) = \dim X - \dim M$.

more generally: we don't have to require a metric on X. Then we set

$$N_x M := \underbrace{T_{i(x)} X \big/ i_*(T_x M)}_{\text{quotient, instead of orthog. complement}}, \qquad NM = \bigcup_{x\in M} N_x M.$$



Ex: for $\underset{\text{unit sphere}}{S^n} \subset \mathbb{R}^{n+1}$, $NS^n = \{(\vec{x}, \alpha\vec{x}) \mid \vec{x} \in S^n, \alpha \in \mathbb{R}\}$ ← iso to a trivial rk=1 bundle!

$\cong\ S^n \times \mathbb{R} = \{(\vec{x}, \alpha)\}$ (with $S^n \subset \mathbb{R}^{n+1}$)

• **Tautological line bundle over $\mathbb{RP}^n$.**

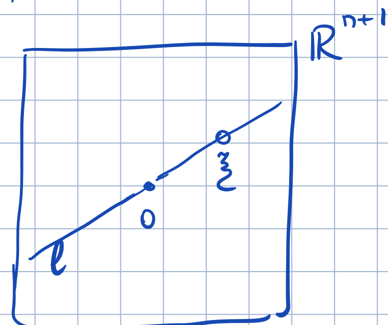
$\mathbb{RP}^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}\} \big/ (x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n)\ \forall \lambda \neq 0$ $= \{\text{lines through } 0 \text{ in } \mathbb{R}^{n+1}\}$

tautological line bundle:

$$\tau = \{(\underset{\text{line through the origin}}{\ell \subset \mathbb{R}^{n+1}}, \underset{\text{point in } \ell}{\xi \in \ell})\}$$

$$\downarrow$$

$$\mathbb{RP}^n = \{\ell \subset \mathbb{R}^{n+1}\}$$



Fiber over a line $\ell = (x_0 : x_1 : \dots : x_n) \in \mathbb{RP}^n$ is the line itself $\{\xi = (\mu\ell_0, \dots, \mu\ell_n) \mid \mu \in \mathbb{R}\}$

Similarly, one has a tautological (complex) line bundle over $\mathbb{CP}^n$,

$$\mathcal{O}(-1) = \tau^{\mathbb{C}} = \{(\ell \subset \mathbb{C}^{n+1}, \xi \in \ell\}$$

$\downarrow$ $\mathbb{CP}^n$; ℓ ← cx. 1-dim subspace

$\mathcal{O}(-1)$ ← another notation

- Tautological bundle over Grassmannians

Grassmannian $Gr_{\mathbb{k}}(k,n) = \{ k\text{-dimensional } \mathbb{k}\text{-linear subspaces of } \mathbb{k}^n \}$, ← a compact smooth manifold, $\dim_{\mathbb{k}} = k(n-k)$ (prove it!)

$\mathbb{k} = \mathbb{R}$ or $\mathbb{C}$

tautological k-plane bundle over $Gr(k,n)$

$$\begin{array}{l} \tau = \{ (\underbrace{W \subset \mathbb{R}^n}_{k\text{-dim subspace}}, \xi \in W) \} \\ \downarrow \\ Gr(k,n) = \{ W \subset \mathbb{R}^n \} \end{array}$$

- and similarly for complex Grassmannians.

- There are many ways to construct new vector bundles out of old ones

① Whitney sum $E_1 \oplus E_2$: $(E_1 \oplus E_2)_x = (E_1)_x \oplus (E_2)_x$ $\qquad t^{E_1\oplus E_2}_{\alpha\beta}(x) = t^{E_1}_{\alpha\beta}(x) \oplus t^{E_2}_{\alpha\beta}(x)$

$\qquad \overset{\shortparallel}{\{ (p_1,p_2) \in E_1 \times E_2 \mid \pi_1(p_1) = \pi_2(p_2) \}} = \text{fiber product } E_1 \times_M E_2$

② tensor product $E_1 \otimes E_2$: $(E_1 \otimes E_2)_x = (E_1)_x \otimes (E_2)_x$ $\qquad t^{E_1\otimes E_2}_{\alpha\beta}(x) = t^{E_1}_{\alpha\beta}(x) \otimes t^{E_2}_{\alpha\beta}(x)$

③ dual E^* : $(E^*)_x = \underbrace{(E_x)^*}_{\text{dual vector space}}$ $\qquad t^{E^*}_{\alpha\beta}(x) = \left(t^{E}_{\alpha\beta}(x) \right)^{-1\,T}$

④ quotient by a sub bundle.

Ex: for $\underset{\text{immersion}}{M \hookrightarrow \mathbb{R}^n}$, normal bundle: $NM = \underbrace{M \times \mathbb{R}^n}_{\text{triv. bundle}} / TM$ - quotient bundle

⑤ symmetric & exterior powers

Ex: $\Lambda^p T^* M$ - bundle of p-forms on M

Ex: metric $\in \Gamma(M, \mathrm{Sym}^2 T^* M)$

notation:
$\Gamma(M,E) = \{ \text{sections } \sigma \text{ of } \begin{array}{c} E \\ \downarrow \\ M \end{array} \}$

Rem: Generally, if we have a functor

$\Phi : (\underbrace{\mathrm{Vect}^{\mathrm{Iso}}}_{\substack{\text{category of v.spaces} \\ \text{with morphisms = isomorphisms}}})^{\times n} \longrightarrow \mathrm{Vect}^{\mathrm{Iso}}$ which is smooth on morphisms, then we have an operation on vector bundles,

$$(E_1, \dots, E_n) \longmapsto \Phi(E_1, \dots, E_n).$$

(examples ①, ②, ③, ⑤ above are of this type)

Pullback of a fiber bundle.

Given a bundle
$$\begin{array}{c} E \leftarrow F \\ \pi \downarrow \\ M \end{array}$$
and a map $f : N \longrightarrow M$

one can form the pullback bundle
$$\begin{array}{c} f^*E \\ \downarrow \pi' \\ N \end{array}$$
where $f^*E := \{(y, p) \in N \times E \mid f(y) = \pi(p)\}$
$\subset N \times E$

proj_1 = bundle projection π',

proj_2 gives the map $F : f^*E \to E$

s.t.
$$\begin{tikzcd} f^*E \arrow[r, "F"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ N \arrow[r, "f"] & M \end{tikzcd}$$

- (Fiber of f^*E over $y \in N$) $= E_{f(y)}$

Theorem Let $\begin{array}{c} E \\ \downarrow \\ M \end{array}$ be a vector bundle and $f_0, f_1 : \boxed{N} \rightrightarrows M$ two homotopic maps.

cpt. Hausdorff, or more generally paracompact [or 1st 2nd countable]

Then the pullback bundles $\begin{array}{c} f_0^*E \\ \downarrow \\ N \end{array}$, $\begin{array}{c} f_1^*E \\ \downarrow \\ N \end{array}$ are isomorphic (as v. bundles).

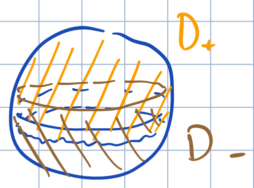
[ref: Hatcher "V.bun. & K-thy", p.20]

Rem: two maps $N \rightrightarrows M$ which are homotopic
C^∞-mfds
are also C^∞-homotopic. - consequence of Whitney Approximation Theorem (see Lee "Intro to Smooth Mfds")

Corollary: every vector bundle over a contractible base is trivial

[$M \underset{f_0 = \text{const. map}}{\overset{f_1 = \mathrm{id}}{\rightrightarrows}} M$, $\begin{array}{c} m \\ \Downarrow \\ M \to x_0 \end{array}$ $\quad f_1^*E = E$, $f_0^*E = M \times E_{x_0}$ - triv. bundle - by Thm, they are isomorphic]

Ex: complex line bundles over $\mathbb{CP}^1$:



trivial over disks D_+, D_- and classified by a transition function $t: D_+ \cap D_- \longrightarrow GL(1,\mathbb{C}) = \mathbb{C}^*$

$D_+ \cap D_- \underset{\text{diffeo}}{\cong} S^1 \times [-\varepsilon, \varepsilon] \underset{\text{homotopic to}}{\sim} S^1$

iso classes of bundles $\overset{1-1}{\longleftrightarrow}$ homotopy classes of maps t

- classified by the winding number $\in \mathbb{Z}$

more generally, rank $= k$ cx bundles over S^n are classified by homotopy classes of maps $S^{n-1} \to GL(k,\mathbb{C})$, i.e. by elements of $\pi_k S^{n-1}$

let $t_1 \sim t_2$

$E_{1,2} = D_+ \times \mathbb{C} \underset{t_{1,2}}{\cup} D_- \times \mathbb{C}$ $\qquad \Phi: E_1 \to E_2$ $\qquad \Phi = 1$ on D_+

$B \times \mathbb{C}$ — equatorial band

$\Phi = t_2 t_1^{-1}$ on $B \subset D_-$
smoothly extended to $D_- \setminus B$
(only possible if $t_2 t_1^{-1}$ is homotopic to const. map!)

Rem: vect. bun. in C^∞ vs. Top category.
(a) For a top v.b. over a C^∞ mfd M, there is an iso. C^∞-bundle
(b) if two C^∞ bundles are iso in Top, they are iso. smoothly
$\}\Leftarrow$ Whitney approximation thm ("for any cont. map $M \xrightarrow{f} N$, there exist a C^∞ map $\tilde{f}: M \to N$ homotopic to f.")

<ref: Ralph Cohen "Bundles, homotopy, and manifolds" p. 19-20>

def A principal G-bundle (with G a topological group) is a fiber bundle with fiber $F = G$

$\pi: P \to M$, with a continuous right action $P \times G \to P$ preserving fibers of π,
(P: total space, M: base)

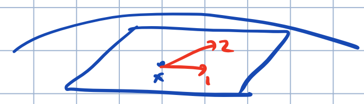
i.e., for $p \in P_x$, $p \cdot g \in P_x$ $\forall g$, and such that for each $x \in M$, G acts on P_x freely and transitively. Also, one has a G-equivariant local trivialization in a nbhd U of each $x \in M$:

$$\pi^{-1}(U)^{\circlearrowleft G} \xrightarrow[\varphi]{} U \times G^{\circlearrowleft G} \qquad \text{i.e. } \varphi(p \cdot g) = \varphi(p) \cdot g$$

- G-orbits of P are the fibers of π
- fibers of π are "G-torsors" - copies of G without a marked unit.
- smooth setting: G - Lie group, all maps in C^∞.

Ex: Frame bundle of a mfd M : FM

fiber over $x = \{$frames in $T_x M\}$.
(ordered bases)



$$\begin{array}{l} FM \circlearrowleft GL(n, \mathbb{R}) \\ \downarrow \qquad\qquad\quad \uparrow \\ M \qquad\qquad\quad \dim M \end{array}$$