LAST TIME

Lemma For any rank $=k$ bundle $\begin{array}{c}E\\\downarrow\\M\end{array}$ over a compact base M,

there exists a map $f: M \to Gr(k,m)$ for m sufficiently large,

s.t. $\boxed{E \underset{\text{iso}}{\simeq} f^* \tau^k}$

Infinite Grassmannian

$\mathbb{R}^\infty := \{$infinite sequences of real numbers $(x_1, x_2, \ldots)$ where all but finitely many x_i are zero$\}$

I.e. $\mathbb{R}^\infty$ is the union of $\underset{(x_1,0,0,\ldots)}{\mathbb{R}^1} \subset \underset{(x_1,x_2,0,\ldots)}{\mathbb{R}^2} \subset \mathbb{R}^3 \subset \cdots$

• Infinite Grassmannian manifold $Gr_{\mathbb{R}}(k,\infty)$ is the set of all k-dimensional linear subspaces of $\mathbb{R}^{\infty}$,
with topology of direct limit of the sequence

$$Gr(k,k) \subset Gr(k,k+1) \subset Gr(k,k+2) \subset \cdots$$

• $Gr(k,\infty)$ is paracompact, (but not compact)

I.e. a subset $U \subset Gr(k,\infty)$ is open / closed iff $U \cap Gr(k,m)$ is open/closed $\forall\, m \geq k$

<u>Ex</u>: $\mathbb{RP}^{\infty} = Gr_{\mathbb{R}}(1,\infty) := \varinjlim \left(\mathbb{RP}^1 \subset \mathbb{RP}^2 \subset \cdots\right)$

• one has the tautological bundle $\underset{rk=k}{}$
$$\begin{array}{c} \tau^k \\ \downarrow \\ Gr(k,\infty) \end{array} \ni (\text{k-plane } \lambda \text{ in } \mathbb{R}^{\infty},\ \text{point } \xi \in \lambda)$$

• <u>Thm</u>: (5.6 in M-S) Any rk=k bundle E over a paracompact base M admits a bundle map
$$\begin{tikzcd} E \arrow[r, "\Phi"] \arrow[d] & \tau^k \arrow[d] \\ M \arrow[r, "f"'] & Gr(k,\infty) \end{tikzcd}$$
inducing linear iso on fibers.

(In particular, $E \underset{iso}{\simeq} f^*\tau^k$)

• <u>Thm</u> (5.7 in M-S) Any two bundle maps $E \underset{f_i}{\xrightarrow{\Phi_i}} \tau^k$, $i=0,1$ (inducing iso on fibers) are bundle-homotopic, i.e. $\exists$ a continuous family of bundle maps
$$\begin{tikzcd} E \arrow[r, "\Phi_t"] \arrow[d] & \tau^k \arrow[d] \\ M \arrow[r, "f_t"'] & Gr(k,\infty) \end{tikzcd}$$
with $t \in [0,1]$, where $\Phi_t|_{t=0,1} = \Phi_{0,1}$, $f_t|_{t=0,1} = f_{0,1}$.

<u>Cor</u> ~~Any rk=k bundle E over a paracompact base M determines a unique homotopy class of maps $f_E : M \to Gr(k,\infty)$.~~

For any paracompact M, there is a bijection

$$\left\{\begin{array}{c} \text{rk=k v.bundles} \\ \text{over } M \end{array}\right\}\Big/ iso \quad \underset{1-1}{\longleftrightarrow} \quad [M, Gr(k,\infty)] = \{\text{maps } M \to Gr(k,\infty)\}\big/ \text{homotopy}$$

$$\begin{array}{c} E \\ \downarrow \\ M \end{array} \longmapsto f_E : M \to Gr(k,\infty)$$
classifying map

$$\begin{array}{c} f^*\tau^k \\ \downarrow \\ M \end{array} \longleftarrow\!\shortmid\ f : M \to Gr(k,\infty)$$

Char. classes as pullbacks of coh. classes of the classifying space $Gr(k,\infty)$

- for Λ a coefficient ring, let $c \in H^i(Gr(k,\infty),\Lambda)$ be some cohomology class.

Then for any v.bun. $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$, we have $\underbrace{f_E^* c}_{\overset{!!}{c(E)}} \in H^i(M,\Lambda)$ – characteristic cohomology class of E determined by c.

- this construction is natural w.r.t. bundle maps.

Conversely, given a natural correspondence $\left(\text{v.bun. } \begin{matrix} E \\ \downarrow \\ M \end{matrix}\right) \mapsto c(E) \in H^i(M,\Lambda)$,

we have $c(E) = f_E^* \, c(\tau^k)$

Thus: $\left\{ \begin{matrix} \text{ring of char. classes} \\ \text{for rk}=k \text{ bundles} \\ \text{with coeffs in } \Lambda \end{matrix} \right\} \underset{\text{can. iso.}}{\simeq} H^\bullet(Gr(k,\infty),\Lambda)$

Cohomology of the Grassmannian.

- cell structure for $Gr(k,m)$

"Stiefel manifold" – principal $GL(k)$-bundle over $Gr(k,m)$

$$\boxed{V(k,m)} = \{(k\text{-plane } \lambda \subset \mathbb{R}^m,\ \text{frame } \vec v_1,\dots,\vec v_k \in \lambda)\}$$
$$= \{\text{lin. indep. } k\text{-tuples } \vec v_1,\dots,\vec v_k \in \mathbb{R}^m\}$$
$$\downarrow \qquad\qquad \downarrow$$
$$Gr(k,m) \ni \lambda = \mathrm{Span}(\vec v_1,\dots,\vec v_k)$$

elem. column operations
= multiplication by an elt of $GL(k)$ on the right

k-tuple of vectors $\vec v_1,\dots,\vec v_k \in \mathbb{R}^m \mapsto \underbrace{(\vec v_1 \cdots \vec v_k)}_{m\times k \text{ matrix}} \longmapsto$ unique reduced (column) echelon form

$$\begin{pmatrix} * & * & & * \\ * & * & & * \\ 1 & 0 & \cdots & 0 \\ 0 & * & & * \\ \vdots & * & \cdots & * \\ \vdots & 1 & 0 & 0 \\ \vdots & 0 & & * \\ 0 & \vdots & & 1 \\ & 0 & & 0 \\ & & & \vdots \\ & & & 0 \end{pmatrix}$$

row σ_1, row σ_2, row σ_k

$1 \le \sigma_1 < \sigma_2 < \dots < \sigma_k \le m$

So:
$$Gr(k,m) = \{\text{lin. indep. } k\text{-tuples in } \mathbb{R}^m\} \big/ GL(k)$$
$$= \bigcup_{\substack{\sigma = (\sigma_1,\dots,\sigma_k) \\ \text{with } 1\le\sigma_1<\dots<\sigma_k\le m}} e_\sigma$$

Schubert symbol

$e_\sigma \cong \mathbb{R}^{d(\sigma)}$, – topological open cell (disk)

$d(\sigma) = (\sigma_1 - 1) + (\sigma_2 - 2) + \dots + (\sigma_k - k)$

total $\#\{\text{cells}\} = \binom{m}{k}$.

Schubert symbol $\sigma \longmapsto$ partition $r = (\sigma_1 - 1) + \dots + (\sigma_k - k)$ of r into at most k integers, each of which is $\leq m-k$.

$\sigma = (\sigma_1 \dots \sigma_k)$ with $d(\sigma) = r$

forgetting zeroes in the beginning

E.g. if $k, m-k \geq r$, then $\#\{r\text{-cells}\} = p(r)$ - number of partitions of r.

Ex: $4 = 1+1+1+1$
$1+1+2$
$2+2$
$1+3$
4
so, $p(4) = 5$

- can take $m \to \infty$. $\rightsquigarrow$ cell decomposition of $Gr(k, \infty)$.

cell cochain complex for $Gr(k, \infty)$ with coeff's in $\mathbb{Z}_2$

$$0 \to C^0(Gr, \mathbb{Z}_2) \longrightarrow C^1 \longrightarrow \dots \longrightarrow C^r \longrightarrow \quad \dots \longrightarrow$$

$C^0 \simeq \mathbb{Z}_2$, $\quad C^r = \mathrm{Span}_{\mathbb{Z}_2}\{e_\sigma\}_{d(\sigma)=r}$

(**) • Fact: all coboundary maps here are zero! (over $\mathbb{Z}_2$).

So, $H^r(Gr, \mathbb{Z}_2) \simeq \mathbb{Z}_2^{\#\text{ partitions of } r \text{ into } \leq k \text{ integers}}$, $\# \text{ partitions of } r \text{ into } \leq k \text{ integers} =: p_k(r)$

Theorem: As a ring, $H^\bullet(Gr(k, \infty), \mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2, \dots, w_k]$

"universal SW classes" = SW classes of $\tau^k \to Gr(k,\infty)$

w_i - generator in degree i.

degree r part generated by the $p_k(r)$ monomials of deg $= r$ in w_i's.

• Assuming SW classes (satisfying the axioms) exist:

- SW classes of $\tau^k \to Gr(k, \infty)$ are algebraically independent (satisfy no relations) in $H^\bullet(Gr, \mathbb{Z}_2)$,

Since for $(\tau^1)^{\times k} \to (\mathbb{RP}^\infty)^{\times k}$, one has a classifying map $f: (\mathbb{RP}^\infty)^{\times k} \to Gr(k, \infty)$

$w = (1+a_1)\cdots(1+a_k)$ total SW class

$\Rightarrow f^* w_i^{(\tau^k)} = \sigma_i(a_1, \dots, a_k)$

σ_i - i-th elem. symmetric poly. in k variables

$\{\sigma_i\}_{i=1}^k$ are alg. independent $\Rightarrow \{w_i\}$ are alg. independent.

lower bound on Betti numbers

(so, $H^\bullet(Gr(k, \infty), \mathbb{Z}_2)$ contains the polynomial algebra $\mathbb{Z}_2[w_1, \dots, w_k]$; by the count of cells in CW decomp, $H^\bullet(Gr, \mathbb{Z}_2)$ is equal to it)

upper bound on Betti numbers

- this implies (**)

(Cor: $f^*: H^\bullet(Gr(k, \infty), \mathbb{Z}_2) \to H^\bullet((\mathbb{RP}^\infty)^k, \mathbb{Z}_2)$ is injective; its image = sym. polynomials in $a_1 \dots a_k$)

- Uniqueness of SW classes

Let $w, \tilde{w}$ be two systems of SW classes (satisfying the axioms)

$w((\mathbb{RP}^\infty)^{\times k}) = (1+a_1)\cdots(1+a_k) = \tilde{w}((\mathbb{RP}^\infty)^{\times k})$

using injectivity of $f^*: H^\cdot(Gr_k) \to H^\cdot((\mathbb{RP}^\infty)^k)$, we have $w(\tau^k) = \tilde{w}(\tau^k)$.

then, for any v.bun. $E \to M \leftarrow$ paracompact, we have $w(E) = f_E^*(\underbrace{w(\tau^k)}_{=\tilde{w}(\tau^k)}) = \tilde{w}(E)$ $\square$