LAST TIME

- $Gr(k,m) = \bigcup_{\sigma = (1 \le \sigma_1 < \dots < \sigma_k \le m)} e_\sigma$ - cell decomposition, $\dim(e_\sigma) = (\sigma_1 - 1) + \dots + (\sigma_k - k)$

- $H^*(Gr(k,\infty), \mathbb{Z}_2) = \mathbb{Z}_2[w_1, \dots, w_k]$ ⟵

  - assuming existence of SW classes, we proved

  (proved alg. independence of $w_1, \dots, w_k$ and that there are no other classes - by dimension counting)

- $(\mathbb{R}P^\infty)^{\times k} \xrightarrow{f} Gr(k,\infty)$

  f ↖ class. map for $(\tau')^{\times k}$

  $f^* : w_i \longmapsto \sigma_i(\underbrace{a_1, \dots, a_k})$

  a_j - generator of $H^1(\mathbb{R}P^\infty, \mathbb{Z}_2)$ ↑ j-th copy

  no relations among σ_i's $\Rightarrow$ w_i alg. indp.

  $\Rightarrow f^*$ is injective

  ($f^* \, p(w_1 \dots w_k) = 0$

  $\,\|$

  $p(\sigma_1 \dots \sigma_k)$ would imply a polyn. relation among σ_j's)

: proved uniqueness of SW classes.

For $Gr(k,\infty)$:

$$w_i(e_{\underbrace{1,1,\dots,1}_{i}}) = 1$$

$$w_i(\text{other cells}) = 0$$

Ex: $Gr(2,\infty)$ cells

$$e_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \vdots & 0 \\ \vdots & \vdots \end{bmatrix}, \quad e_1 = \begin{bmatrix} 1 & 0 \\ 0 & * \\ \vdots & 1 \\ \vdots & 0 \\ \vdots & \vdots \end{bmatrix}, \quad e_{11} = \begin{bmatrix} * & * \\ 1 & 0 \\ 0 & 1 \\ \vdots & 0 \\ \vdots & \vdots \end{bmatrix}, \quad e_2 = \begin{bmatrix} 1 & 0 \\ 0 & * \\ \vdots & * \\ \vdots & 1 \\ & 0 \\ & \vdots \end{bmatrix}$$

w_1 is nontriv. here (e_1)

w_1^2 (e_2)

w_2 is nontriv here (e_{11})

$$e_3 = \begin{bmatrix} 1 & 0 \\ 0 & * \\ \vdots & * \\ \vdots & * \\ \vdots & 1 \\ & 0 \\ & \vdots \end{bmatrix}, \quad e_{12} = \begin{bmatrix} * & * \\ 1 & 0 \\ 0 & * \\ \vdots & 1 \\ \vdots & 0 \\ \vdots & \vdots \end{bmatrix}$$

w_1^3 (e_3)

$w_1 w_2$ (e_{12})

Existence of S-W classes - construction of w_i via Thom isomorphism and Steenrod squares Sq^i.
(M-S, § 8)

- **Thom isomorphism** For a v.bun. of rank k $\;\begin{matrix} E \\ \downarrow \pi \\ M \end{matrix}$, let $E_0 := E \setminus$ zero-section.

remark: $H^i(\mathbb{R}^k, \mathbb{R}^k \setminus \{0\}) = \begin{cases} 0, & i<k \\ \mathbb{Z}_2, & i=k \end{cases}$

(#) ← $\mathbb{Z}_2$-coeffs everywhere

Thm a) $H^i(E, E_0) = 0$ for $i < k$

b) $H^k(E, E_0)$ contains a unique class u ("Thom class") s.t. $u|_{E_x, E_x\setminus\{0\}} \in H^k(E_x, E_x \setminus \{0\})$ (restriction to a fiber) is the unique nonzero class.

Furthermore, one has an isomorphism $H^i(E) \to H^{i+k}(E, E_0)$, $i \geq 0$.
$\alpha \longmapsto \alpha \cup u$

def: Composition of isos

$$\begin{tikzcd} H^i(M) \arrow[r, "\pi^*"', "\text{iso since } E \text{ retracts onto } M"] \arrow[rr, bend right, dashed, "\varphi"'] & H^i(E) \arrow[r, "\cup u"'] & H^{i+k}(E, E_0) \end{tikzcd}$$

is called Thom isomorphism.

- **Steenrod squaring operations** in $H^*(-; \mathbb{Z}_2)$

-characterized by the following properties

(1) for $X \supset Y$ a pair of top. spaces and $i, k \geq 0$, there is an additive homomorphism

$$Sq^k : H^i(X,Y) \to H^{i+k}(X,Y)$$

(2) **Naturality** If $f : (X,Y) \to (X',Y')$ then $Sq^k \circ f^* = f^* \circ Sq^k$

(3) If $a \in H^i(X,Y)$, then
- $Sq^0(a) = a$
- $Sq^i(a) = a \cup a$
- $Sq^k(a) = 0$ for $k > i$

(thus, the most interesting operations are Sq^k with $0 < k < i$)

(4) Cartan's formula $\quad Sq^k(a \cup b) = \sum_{r+s=k} Sq^r(a) \cup Sq^s(b)$.

(one can introduce the total squaring operation $Sq(a) = a + Sq^1(a) + \dots + Sq^i(a)$ for $a \in H^i(X,Y)$
then: $Sq(a \cup b) = Sq(a) \cup Sq(b)$)

In terms of Thom iso and Sq-operations, one can define SW classes by

$$w_i(E) := \varphi^{-1} \underbrace{Sq^i \underbrace{\varphi(\underset{\overset{\cap}{H^0(M)}}{1})}_{u \in H^k(E,E_0)\ \text{- Thom class}}}_{\in H^{k+i}(E,E_0)}$$

~ can verify the axiom of SW classes from this construction (M-S §8)

Euler class

Let $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ be an oriented vector bundle. (fibers are oriented in a loc. trivial way)
<=> transition functions are compatible with stand. orientation

Orientation <=> choice of a generator in $\underbrace{H^k(E_x, E_x \setminus \{0\}; \mathbb{Z})}_{H^k(S^k, pt; \mathbb{Z}) \simeq \mathbb{Z}}$ for all fibers E_x.

Thm for $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ rk k, oriented, a) $H^i(E, E_0; \mathbb{Z}) = 0$, $i < k$

b) $H^k(E, E_0; \mathbb{Z})$ contains a unique class u s.t. $u|_{E_x, E_x \setminus \{0\}} \in H^k(E_x, E_x \setminus \{0\}; \mathbb{Z})$
= orientation class of the fiber.

One has an iso $H^i(E; \mathbb{Z}) \to H^{i+k}(E, E_0; \mathbb{Z})$, $i \geq 0$

$\alpha \longmapsto \alpha \cup u$

Thus, one has iso

$$\begin{tikzcd} H^i(M;\mathbb{Z}) \arrow[r, "\pi^*"] \arrow[rr, bend right, dashed, "\varphi"'] & H^i(E;\mathbb{Z}) \arrow[r, "\cup u"] & H^{i+k}(E, E_0; \mathbb{Z}) \end{tikzcd}$$

- Thom isomorphism

One has

$$\begin{tikzcd} H^k(E, E_0; \mathbb{Z}) \arrow[r, "\text{pullback by } (E,\emptyset) \hookrightarrow (E,E_0)"] & H^k(E; \mathbb{Z}) \arrow[r, "(\pi^*)^{-1}"'] & H^k(M, \mathbb{Z}) \end{tikzcd}$$

$$u \longmapsto \overset{\overset{\cup}{}}{e}(E) \quad \text{- "Euler class"}$$

Properties (1) Naturality: for

$$\begin{tikzcd} E \arrow[r, "\Phi"] \arrow[d] & E' \arrow[d] \\ M \arrow[r, "f"'] & M' \end{tikzcd}$$

, a bundle map inducing orientation preserving isos on fibers,

$e(E) = f^* e(E')$.

Cor: if $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ is a trivial bundle of rk > 0, $e(E) = 0$

(2 If $\bar{E}$ - bundle E with reversed fiber orientation, then $e(\bar{E}) = -e(E)$

(3) If rk E is odd then $e(E)=0$

[For rk E odd, the bundle automorphism $(x,\xi) \overset{(f,\phi)}{\longmapsto} (x,-\xi)$ is or-reversing

$\Rightarrow$ $e(E) = -\underbrace{f}_{id}{}^{*}e(E) = -e(E)$ ⌟

(4) The homom. $H^k(M,\mathbb{Z}) \longrightarrow H^k(M,\mathbb{Z}_2)$ (induced by $\mathbb{Z} \xrightarrow{mod\,2} \mathbb{Z}_2$ in coefficients)

carries $e(E)$ to $w_k(E)$

(5) $e(E\oplus E') = e(E) \cup e(E')$

$e(E\times E') = e(E) \otimes e(E')$

(6) If an oriented v.bun $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ possesses a nowhere-zero section σ, then $e(E)=0$.

[$E = \underset{\substack{\text{span}(\sigma)\\ \text{-trivial}}}{\overset{}{\varepsilon}} \oplus \underset{\text{using some fiber metric}}{\varepsilon^\perp} \cong \underline{\mathbb{R}} \oplus \varepsilon^\perp \Rightarrow e(E) = 0 \cdot e(\varepsilon^\perp) = 0$]

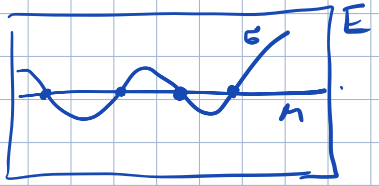
(7) More precise statement: ^(assume M compact, oriented) if σ is a "generic" section of E, i.e. with

$\underset{E}{\text{graph}(\sigma)}$ intersecting the zero-section $z(E)$ transversely.

Z = Zero-locus of σ (= the oriented intersection $\text{graph}(\sigma) \cap z(E)$) $\in \overset{\text{cycles}}{Z_{n-k}}(M,\mathbb{Z})$

represents a class in $H_{n-k}(M,\mathbb{Z})$

Its Poincaré dual is $e(E)$.



(8) For M an oriented n-manifold,

$$\langle \underset{H^n(M,\mathbb{Z})}{\overset{}{e(TM)}}, \underset{H_n(M,\mathbb{Z})}{[M]} \rangle = \chi(M)$$ the Euler characteristic

Rem a char. class is called "stable" if it doesn't change under $E \to E \oplus \overset{\text{triv. bundle}}{\underline{\mathbb{R}}}$.

SW Classes w_i are stable ; Euler class is unstable.

Chern classes

Let
$$\begin{tikzcd} E \arrow[d] & \mathbb{C}^k \arrow[l] \\ M & \end{tikzcd}$$
be a complex vector bundle of rk = k.

Rem: • E has an underlying $\overset{rk=2k}{\text{real}}$ vector bundle
$$\begin{tikzcd} E_{\mathbb{R}} \arrow[d] & \mathbb{R}^{2k} \arrow[l] \\ M & \end{tikzcd}$$

Complex structure on $\overset{\text{a real v.b.}}{E_{\mathbb{R}}}$ = choice of $J_x \in \mathrm{End}(E_{\mathbb{R}})_x$ with $J_x^2 = -\mathrm{Id}$
depending continuously/smoothly on $x \in M$.

$J : \underset{\underset{E}{\cap}}{v} \mapsto iv$

- a cx v.bun. E has a canonical orientation given by the class of 2k-tuple of vectors in E_x
$(v_1, iv_1, v_2, iv_2, \dots, v_k, iv_k)$, for $v_1, \dots, v_k$ any k-tuple of vectors lin. indep over $\mathbb{C}$.

- in a cx. v.b. one can always choose a Hermitian metric in fibers $\langle u,v \rangle \in \mathbb{C}$ for $u, v \in E_x$
  - $\mathbb{C}$-linear as a function of v $\quad \langle u, \lambda v \rangle = \lambda \langle u,v \rangle$
  - $\mathbb{C}$-anti-linear as a function of u $\quad \langle \lambda u, v \rangle = \bar{\lambda} \langle u,v \rangle$
  - $\langle v,v \rangle > 0$ for $v \neq 0$
  - $\langle u,v \rangle = \overline{\langle v,u \rangle}$ (follows from the previous axioms)

- Hermitian metric on E $\xleftrightarrow[1-1]{}$ Euclidean metric on $E_{\mathbb{R}}$ satisfying $\|iv\| = \|v\|$

Construction of Chern classes

For
$$\begin{tikzcd} E \arrow[d, "\pi"] \\ M \end{tikzcd}$$
a rk = k cx. v.bun., choose a Herm. metric $\langle , \rangle$.

$$\begin{tikzcd} SE = \{(x, v \in E_x) \mid \langle v,v \rangle = 1\} \arrow[d, "\pi"] \\ M \end{tikzcd}$$
- unit sphere bundle

Let
$$\begin{tikzcd} \check{E} \arrow[d] \\ SE \end{tikzcd}$$
be the rk = k-1 cx. v.bun over SE with fiber $(\check{E})_{x,v} = \{u \in E_x \mid \langle u,v \rangle = 0\}$
$= (\mathrm{span}_{\mathbb{C}}\{v\})^{\perp} \subset E_x$

Fact: for $i \leqslant 2k-1$, $\pi^* : H^i(M) \longrightarrow H^i(SE)$ is an isomorphism

(follows from Gysin sequence

$$\begin{tikzcd}
\cdots \arrow[r] & \underset{\overset{0}{\parallel}\ \text{for } i<2k-1}{H^{i-2k}(M)} \arrow[r, "\cup e"'] \arrow[d, "\wr"', "\text{Thom iso}"] & H^i(M) \arrow[r, "\pi^*"'] \arrow[d, equal] & H^i(SE) \arrow[r] \arrow[d, equal] & \underset{=0}{H^{i-2k+1}(M)} \arrow[r] \arrow[d, "\wr"] & \cdots \\
\cdots \arrow[r] & H^i(E,E_0) \arrow[r] & H^i(E) \arrow[r] & H^i(E_0) \arrow[r] & H^{i+1}(E,E_0) \arrow[r] & \cdots
\end{tikzcd}$$

)

def Chern classes $c_i(E) \in H^{2i}(M;\mathbb{Z})$ are defined by induction in $rk(E)$:

- Top Chern class $\boxed{c_k(E) := e(E_{\mathbb{R}})}$ - the Euler class.
- For $i<k$, $\boxed{c_i(E) := (\pi^*)^{-1} \underbrace{c_i(\check{E})}_{\in}}$

$$H^{2i}(M) \longleftarrow H^{2i}(SE)$$

- For $i>k$, $c_i(E) = 0$.